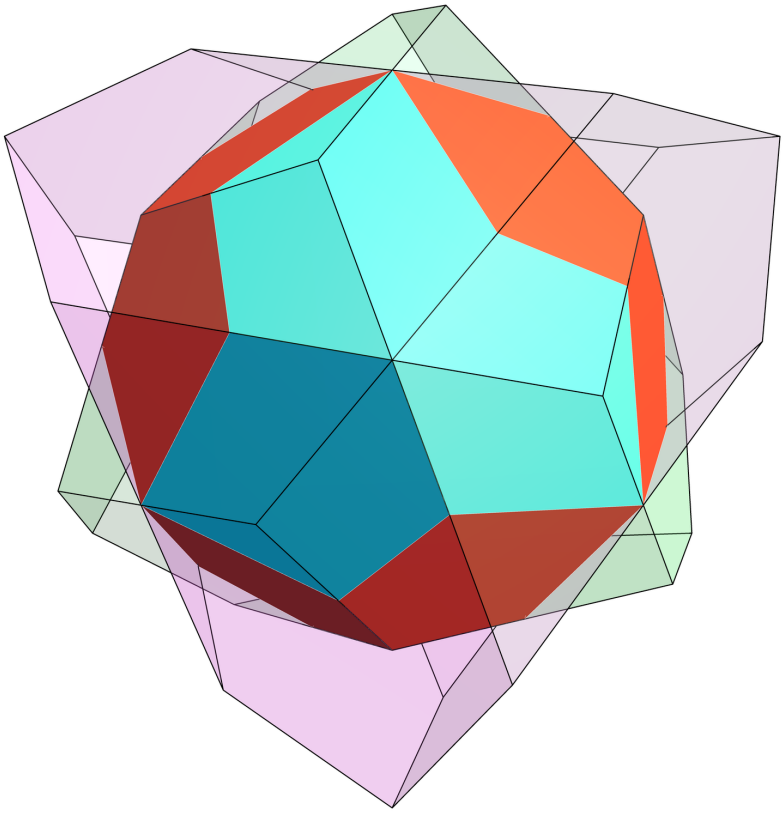
Slidables



Autumn 2026

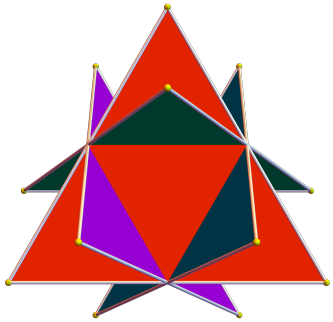
Matthias Weber

This work is licensed under a **Creative Commons Attribution-NonCommercial-Share-Alike 4.0 International License**.

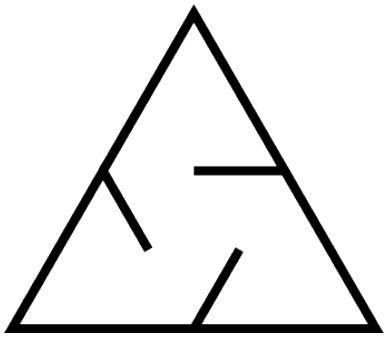
Contents

1	Introduction	2
2	Stella Octangula	2
3	Small Stellated Dodecahedron	3
4	Regular Polyhedra	4
5	Stellated Cuboctahedron	5
6	Stellations	5
7	Small Triambic Icosahedron	6
8	Escher's Solid	6
9	Stellated Icosidodecahedron	7
10	Stellated Triacontahedron	9
11	Duality	11
12	Cuboctahedron and Dual	12
13	Truncated Octahedron and Dual	12
14	Rhombicuboctahedron and Dual	13
15	Snub Cube and Dual	13
16	Truncated Icosahedron and Dual	14
17	What is possible?	15
18	Two Pyritohedra	15
19	Prism and Dual	16
20	Antiprism and Dual	16
21	Rhombicosidodecahedron with Dual	17
22	Two Tetartoids	18
23	Templates	20

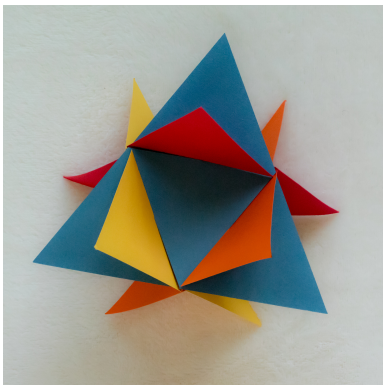
1 Introduction



This booklet contains puzzles for beginners and experts that use simple planar shapes that I call *slidables* to build complex polyhedral shapes.

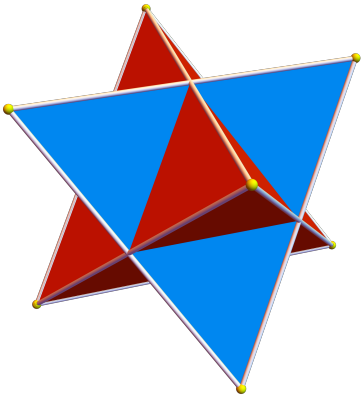


To illustrate what we are going to do, cut out four triangles, as shown below, from cardstock, making sure to cut carefully along the three slits.
The puzzle is to slide these triangles into each other to build the following simple star:

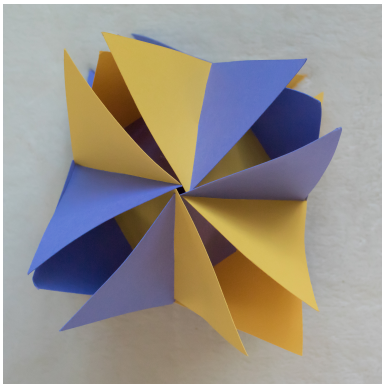


While it is possible to use scissors to cut out the templates, I recommend using a die cutter to have them cut automatically.

2 Stella Octangula

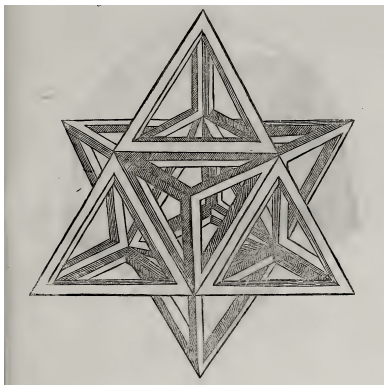


With eight of the same slit triangles, you can build the following shape, which was named *Stella Octangula* by JOHANNES KEPLER in his Harmonice Mundi from 1619.

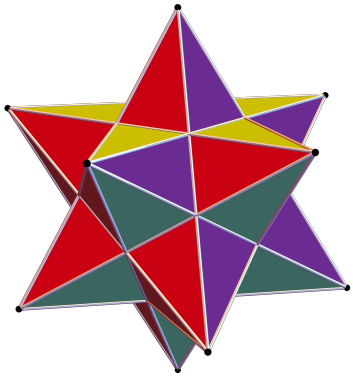


It appeared already 1509

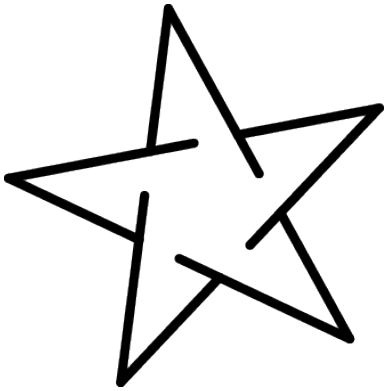
in LUCA PACIOLI'S De Divina Proportione as an illustration by LEONARDO DA VINCI. It can be considered as the compound of two tetrahedra or as a stellated octahedron.



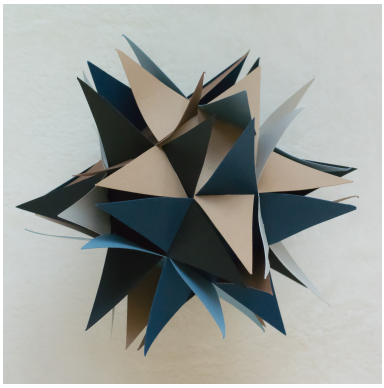
3 Small Stellated Dodecahedron



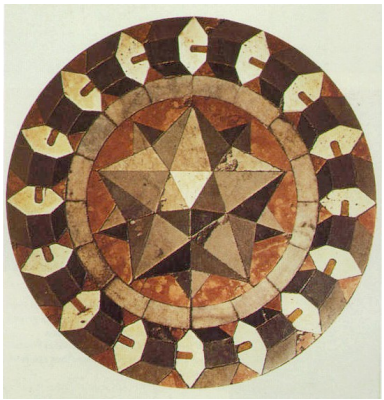
Using 12 copies of the slit pentagrams below, one can build the Small Stellated Dodecahedron.



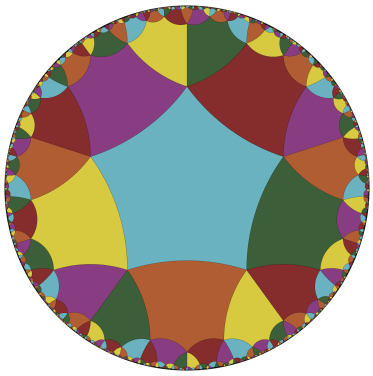
This requires a bit of patience. Start with one pentagram, and insert five pentagrams successively in all of its slits, thereby also linking the inserted pentagrams together. Then insert another five pentagrams into neighboring pairs of the first ring of pentagrams, again linking the pentagrams from the new ring together. Finally, insert the twelfth pentagram into the free slits of the pentagrams from the second ring.
The last steps require some delicate bending of the pentagrams and a careful adjustment at the end.



For a simple choice of colors, use six different colors and two pentagrams of each color. Construct the model so that opposite pentagrams have the same color.
The Small Stellated Dodecahedron was described by JOHANNES KEPLER in his Harmonice Mundi.



However, there is a floor mosaic in St Mark's Basilica in Venice, which is attributed to PAOLO UCCELLO and dates from approximately 1430.



The Small Stellated Dodecahedron has the same Euclidean symmetry group of orientation preserving symmetries as the dodecahedron, namely the alternating group A_5, a group with 60 elements.

In 1877, FELIX KLEIN interpreted each pentagram as a branched version of a regular pentagon. Then the small stellated dodecahedron becomes a hyperbolic Riemann surface of genus 4 that is tiled by 12 regular hyperbolic pentagons, with five pentagons at each vertex. In this version, the surface is even more symmetric than the Euclidean polyhedron.

As a Riemann surface, it is isomorphic to Bing's curve, which is the complete intersection in $\mathbb{P}^4$ of the three hypersurfaces

$$z_1 + ... + z_5 = 0$$
$$z_1^2 + ... + z_5^2 = 0$$
$$z_1^3 + ... + z_5^3 = 0\,.$$

Bing's curve is clearly invariant under the symmetric group S_5, which has 120 elements, and acts by permuting the coordinates.

4 Regular Polyhedra

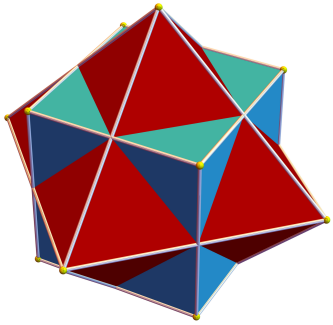
Polyhedra with many symmetries have attracted mathematicians and artists for millennia. The most symmetric polyhedra are the five Platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. All faces are required to be congruent, regular polygons, and symmetries of space can be used to realize all possible face congruences.

Less restrictive is the class of Archimedean solids: Here, one allows regular polygons with different numbers of vertices, and requires that all vertices look alike: For any pair of vertices, there needs to be a symmetry of space that maps the polyhedron to itself and the first vertex to the second vertex. There are 13 Archimedean solids (including the Platonic solids), if one excludes the two infinite families of regular prisms and antiprisms.

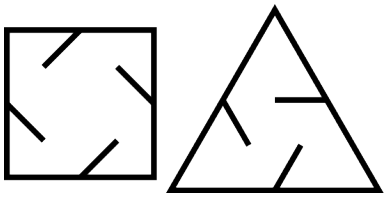
An even larger class is the set of 92 Johnson solids, convex polyhedra where all faces are regular polygons, but no further regularity is required.

Little is known about monohedral polyhedra where all faces have to be congruent, without any symmetry assumptions. We will encounter several polyhedra of each type.

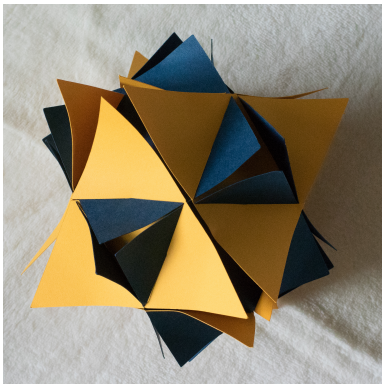
5 Stellated Cuboctahedron



The Cuboctahedron is an Archimedean solid. It can be constructed as the rectification of a cube (or an octahedron), and has eight triangular and six square faces. These faces are obtained by cutting off the vertices of the cube (or octahedron) along planes that pass through the midpoints of the edges that are adjacent to the vertex that is cut off.
Our stellation replaces the triangles and squares with larger triangles and squares, as shown.
This model is also a compound of a cube and an octahedron.
It requires two different types of slidables: Eight triangles (of the same kind we have already met), and six squares. These slidables must be compatible in size so that all slits have the same length.



This model is a bit easier to assemble than the previous example. In the first steps, it falls apart easily, but becomes quite sturdy once complete.



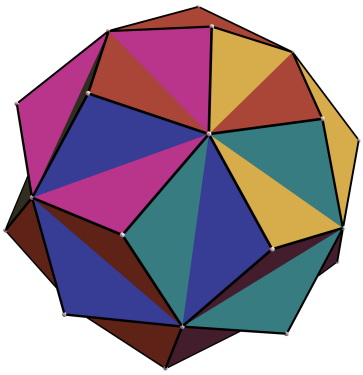
A natural color choice is to use two different colors, one for the triangles and the other for the squares.

6 Stellations

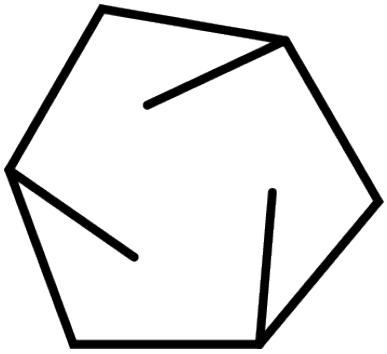
A *stellation* of a polyhedron is obtained by replacing each face by a polygon in the same plane that has as its edges being contained in the intersection of this plane with other face planes of the original polyhedron.
For instance, in the Small Stellated Dodecahedron, one replaces each pentagon by a pentagram.

Stellations are numerous and often very complicated. The simpler stellations are often also obtainable as compounds. However, there are exceptions:
The Small Stellated Dodecahedron, the Small Triambic Icosahedron, Escher's solid, and the Stellated Triacontahedron are stellations that are not easily realizable as compounds.

7 Small Triambic Icosahedron

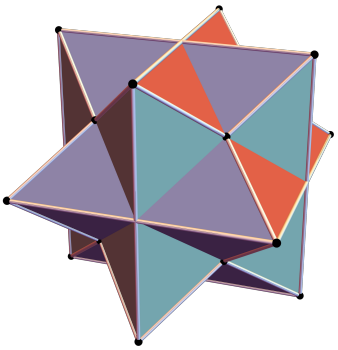


The Icosahedron is a Platonic solid with 20 triangular faces. It has 59 different stellations, most of them very intricate.
One of them is the Small Triambic Icosahedron, in which the triangles of the Icosahedron are replaced by the irregular hexagons below.

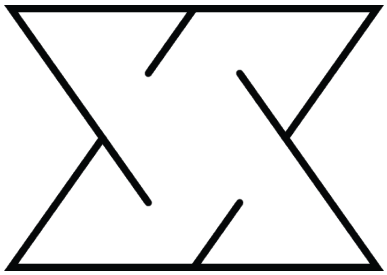


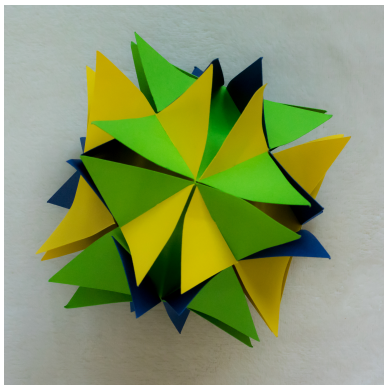
You will need 20 of them. A symmetric color choice uses five colors, with four hexagons of each. These should be placed so that the midpoints of the hexagons of the same color are at the vertices of a regular tetrahedron. This placement is *chiral*; it has no reflectional symmetries but instead full tetrahedral symmetry.

8 Escher's Solid

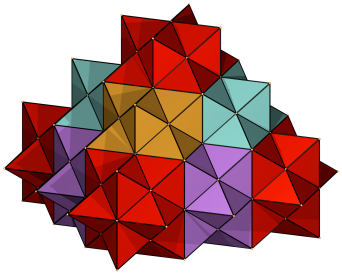


This is the first stellation of the Rhombic Dodecahedron, also called *Escher's Solid*. It appears in M. C. ESCHER'S Waterfall woodcut. You will need 12 of the nonconvex hexagons below.

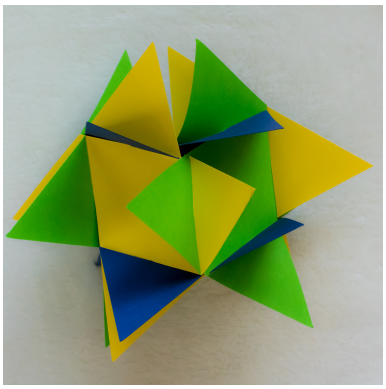




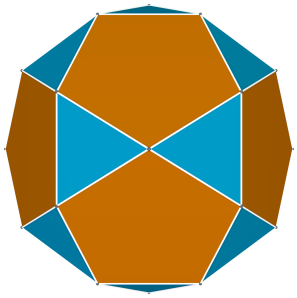
Remarkably, this intricate polyhedron tiles space:



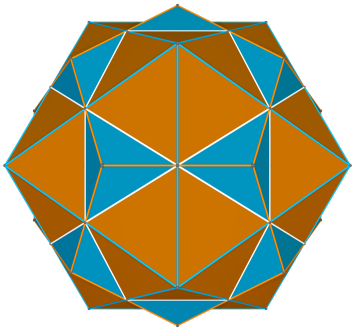
Six of these hexagons can be used to make a simpler model, a stellation of a rhomboid:



9 Stellated Icosidodecahedron

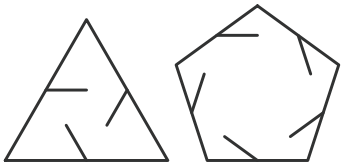


The Icosidodecahedron is an Archimedean solid that can be obtained from either the Dodecahedron or the Icosahedron through *rectification*. This is completely analogous to the Cuboctahedron, which is obtained as a rectification of the Cube or the Octahedron. Here, it means to cut off pyramids at all vertices (pentagonal pyramids for the Icosahedron, triangular pyramids for the Dodecahedron) that have as a base a regular polygon (a pentagon for the Icosahedron, a triangle for the Dodecahedron) that has its vertices at the midpoints of all edges adjacent to the vertex that is being cut off.



The first stellation of the Icosidodecahedron is also a compound of the Dodecahedron and the Icosahedron.
To make this model, you

will need 20 triangles and 12 pentagons as shown. Again, the two types of polygons need to have the correct size so that the slits have the same lengths.



We start by sliding five triangles into one pentagon.

Then we add five pentagons between each pair of adjacent triangles.

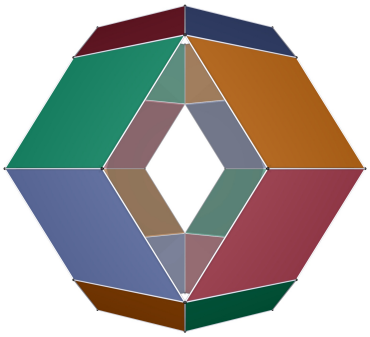
Next come another five triangles between each pair of adjacent pentagons.

We have now finished half of the model. This would already make a nice backyard dome.
You can make another half and try to attach the two halves, but I think it is easier to keep going. This next step is a little tricky. To prevent the polygons from falling out, it is best to add a ring of alternating pentagons and triangles. When done, it will look like this, looking inside the model:

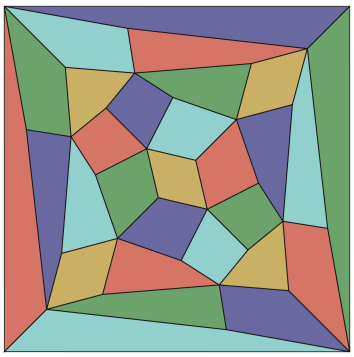
The last two steps (first add five more triangles, and then one more pentagon) are straightforward, but still tricky because you have to insert the new polygons in four or five slits essentially at once.

10 Stellated Triacontahedron

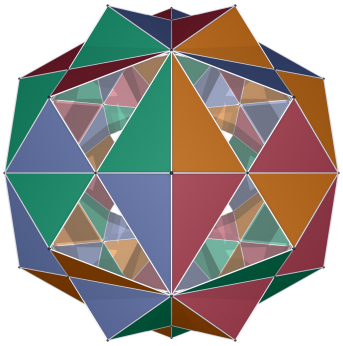
The Rhombic Triacontahedron is a zonohedron (meaning it is tiled with rhombi) with 30 golden rhombi as faces. There are two types of vertices, 12 with valency 5, and 20 with valency 3. In the image below, the faces are colored with five colors, one of which is transparent.



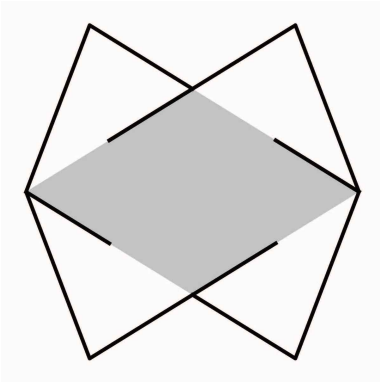
The face coloring is made more apparent in the planar map of this polyhedron below. This will be helpful for our next model.



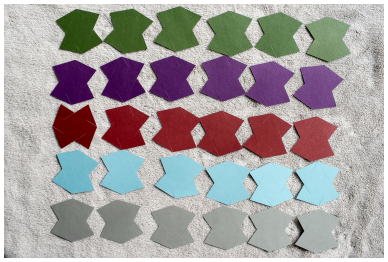
We are going to make a paper model of one of the 358,833,072 stellations of the Rhombic Triacontahedron. This number comes from George Hart's highly inspiring **Virtual Polyedra**.



In this stellation, each golden (or rather, gray) rhombus of the original polyhedron is replaced by a non-convex octagon in the same plane.



The picture above serves as a template. You will need 30 of them, cut along the dark black edges. Choose five color and print six of each color:



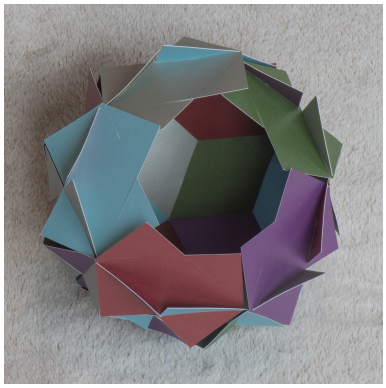
Now assemble five of them, one of each color, around a vertex. Note that there are different ways to put two together; make sure that the original golden rhombi always have acute vertices meeting acute vertices. This produces the first layer.

The next layer of five templates addresses the 3-valent vertices of the first layer. Here, the coloring starts to play a role.

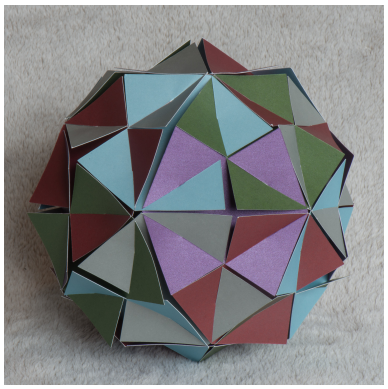
The third layer is the most difficult, because you have to add 10 templates, creating vertices of valency 5 again. The next image shows how to pick the colors to maintain consistency.

Below is the inside of the completed third layer.

Two more layers to go. Layer 4 is easy:



The last layer is again tricky, but only because it gets tight. Here is the finished model. It is pretty sturdy.



11 Duality

There is a notion of duality for polyhedra: For instance, the cube and the octahedron are dual to each other. This notion is at first purely combinatorial: Each polyhedron defines an abstract map of faces, edges, and vertices. For any map, one can define the dual map. The vertices/faces of the dual map are the faces/vertices of the original map. An edge connects two vertices of the dual map if the faces they correspond to in the original map are adjacent.

The challenge is to realize the dual map again as a polyhedron. This requires choosing the coordinates of the vertices of the dual polyhedron so that those vertices that belong to a single face become coplanar. A plausible choice for the new vertices is the centers of the old faces, but the coplanarity condition might generally not be satisfied. As we are interested in making models of compounds of one polyhedron with its dual, we would like to extrude the new vertices so that an edge and its dual edge intersect.

We will now show that even with this additional condition, one can still expect a 6-dimensional parameter space of realizations of dual polyhedra, for any given polyhedron. To see this, let v, e, and f denote the number of vertices, edges, and faces of the original polyhedron. Then we have $3f$ real parameters for the coordinates of the vertices of the dual polyhedron. The requirement that an edge and its dual are coplanar gives us e equations. Now, denote by n_i the valency of vertex i of the original polyhedron. This is the number of vertices of the dual face i. So that face i of the dual polyhedron is planar, we have $n_i - 3$ equations, for a total of

$$\sum_{i=1}^{v} (n_i - 3) = 2e - 3v$$

planarity equations. Thus, altogether, we have $3(e - v)$ equations. Euler's formula states that for a map on a spherical polyhedron we always have

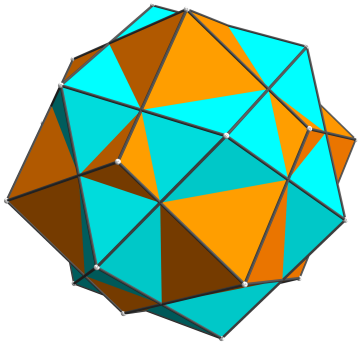
$$v - e + f = 2 \ .$$

Thus, we expect to have

$$3f - (3e - 3v) = 6$$

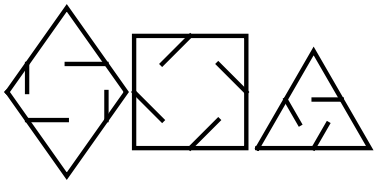
free parameters. The equations are quadratic, so one can say little about the nature of the solution set. Even for a tetrahedron, the solution space is very complicated, and indeed 6-dimensional.

12 Cuboctahedron and Dual

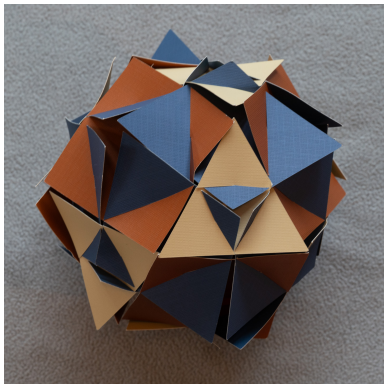


The shape we will create is a compound of the Cuboctahedron and its dual, the Rhombic Dodecahedron.
For this compound, we will need

- 8 triangles;
- 6 squares;
- 12 rhombi.

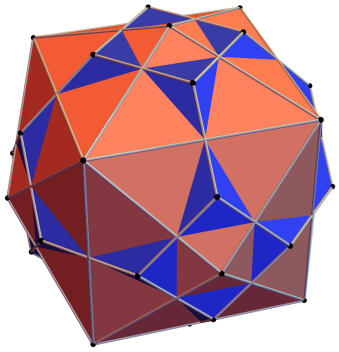


It is fairly easy to build, just keep in mind that the acute vertices of the rhombus belong to the squares and the obtuse vertices to the triangles.



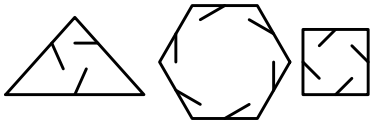
The sizes of the shapes are adjusted so that the corresponding slits have equal lengths.

13 Truncated Octahedron and Dual

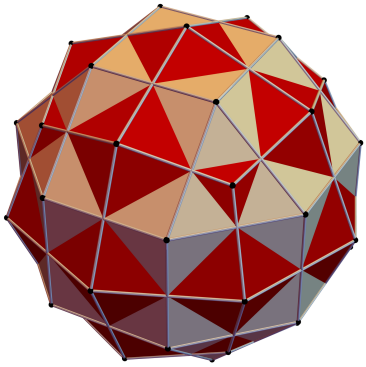


The dual of the Truncated Octahedron is called the Tetrakis Hexahedron.
This model needs three different shapes:

- 12+12 triangles
- 6 squares
- 8 hexagons



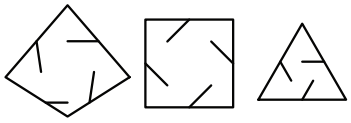
14 Rhombicuboctahedron and Dual



The dual of the Rhombicuboctahedron is the Deltoidal Icositetrahedron. This model needs three different shapes:

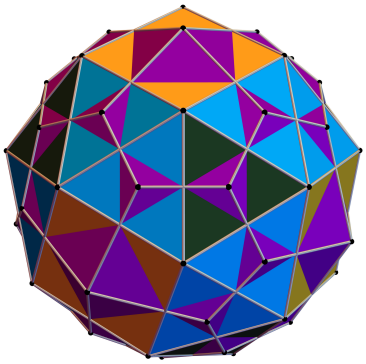
- 8 triangles
- 6+12 squares
- 24 Deltoids

I suggest using six squares of one color and 12 squares of another color. The first six squares correspond to the faces of the cube, and the remaining 12 correspond to the edges. This is, of course, not important; you can use any colors you like. I just find it aesthetically appealing to emphasize the symmetries, and simultaneously, the different colors help to keep track of one's progress.

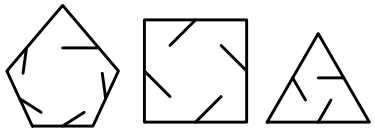


The elongated square gyrobicupola is very similar polyhedron that can be built with the same slidables, you just have to rotate one of the domes (consisting of a square face and its eight adjacent faces) by 45°.

15 Snub Cube and Dual



The Snub Cube is a chiral Archimedean solid, meaning that it is not congruent (by motion) to its mirror image.
The dual of the Snub Cube is called the Pentagonal Icositetrahedron.



The Snub Cube is a curious Archimedean solid. The cyclic permutations of the eight points

$$\left(\pm 1, \pm \frac{1}{\tau}, \pm \tau\right)$$

are the 24 vertices of the Snub Cube. Here τ denotes the *Tribonacci constant*, the unique real root of the cubic equation

$$\tau^3 = \tau^2 + \tau + 1$$

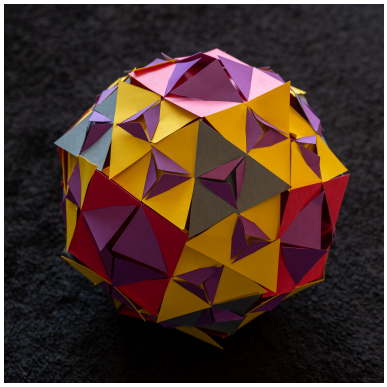
Explicitly, one has the completely useless expression

$$\tau = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right).$$

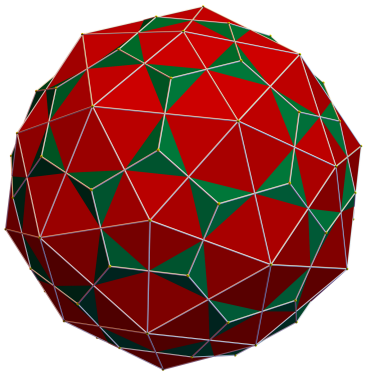
This model requires three different slidables:

- 24+8 triangles
- 6 squares
- 24 pentoids

Note that a pentoid slides at its acute vertex into a square. I suggest using 24 of the 32 triangles as neighbors of the squares, and assigning a different color to the remaining eight.



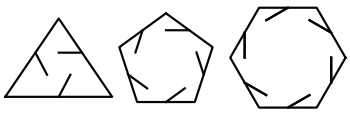
16 Truncated Icosahedron and Dual



Truncation of the icosahedron changes the 20 triangular faces into regular hexagons and creates a regular pentagon for each of the 12 vertices.
Its dual is called the Pentakis dodecahedron.
This model needs three different shapes:

- 60 isosceles triangles
- 12 pentagons
- 20 hexagons

The Truncated Icosahedron is also the standard pattern on a soccer ball.



The triangles are almost regular. Make sure to slide the more obtuse corner into the pentagons, and the others into the pentagons. The slits have slightly different lengths.

17 What is possible?

I have a few questions about the possibilities that this concept of using slidables to construct shapes offers.

For a shape to hold together, all faces must intersect at least three other faces, meaning that all slidables should have at least three slits. On the other hand, there can't be too many slits, because they are not allowed to intersect. Most slidables have as many slits as edges, but there are exceptions.

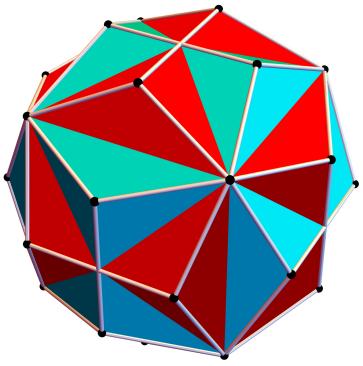
The easiest class of shapes that one can build is that of a compound of a shape and its dual. I have not found an example of a compound involving more than two shapes that I could build.

Then there are a few stellations (and partial stellations) that one can build, but here I find it even harder to predict what is possible.

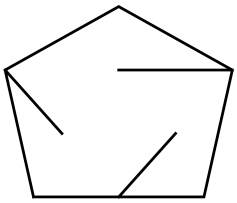
What topological types of polyhedra can one make? For instance, can one make a torus with slidables? The vertices of negative curvature will undoubtedly cause problems.

Finally, I have a mechanical question: Building a model seems to require bending the paper occasionally. Is that really necessary? In other words, if one has slidables made out of some rigid material (thin sheets of metal, for instance), is there still a way to assemble the shape, maybe in the same way some burr puzzles require all pieces to move simultaneously?

18 Two Pyritohedra

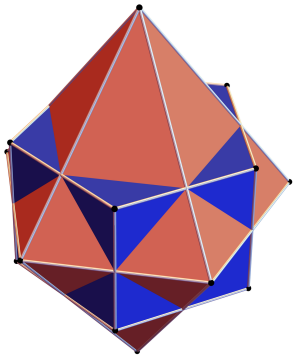


A pyritohedron is a special kind of irregular dodecahedron. To build this model, you will need 24 copies of the slidable below. I suggest cutting 12 each in two different colors to distinguish the two pyritohedra.



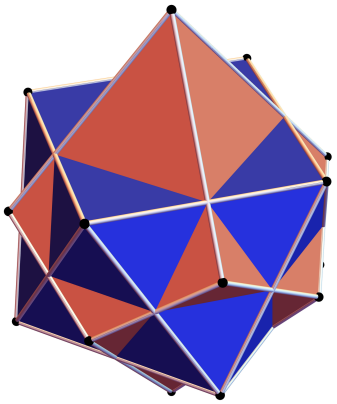
Pyritohedra are a 1-parameter family of dodecahedra with congruent pentagonal faces. The pentagons have a reflectional symmetry about one axis, but are usually not regular. For one particular parameter value, one obtains the crystal structure of pyritohedral pyrite, hence the name. We construct a compound of two such pyritohedra, which also occurs in nature as twinned crystals.

19 Prism and Dual



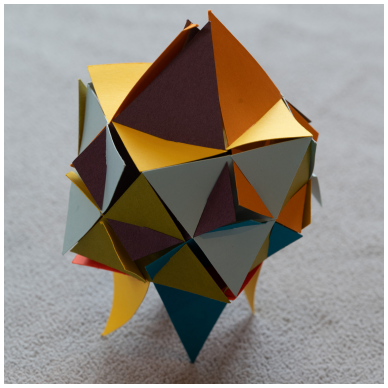
A prism over a regular polygon has two types of faces: The regular polygon appears twice, as the top and bottom, and they are joined by as many squares as the polygon has edges. The best-known example is the prism over a square, also known as the cube.
For any $n \geq 3$, one can make a model of a compound of a regular prism and its dual, a bi-pyramid. For $n = 4$, one obtains the compound of cube and octahedron. The model for $n = 5$ is shown above. I have not made slidables for this model yet, so you are on your own if you want to build it.

20 Antiprism and Dual

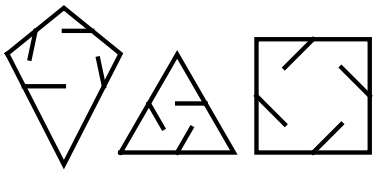


An antiprism over a regular polygon has two types of faces: The regular polygon appears twice, as the top

and bottom, and they are joined by twice as many triangles as the polygon has edges. The best-known example is the prism over a triangle, also known as the octahedron.
For any $n \geq 3$, one can make a model of a compound of a regular antiprism and its dual, an n-gonal trapezohedron. For $n = 3$, one obtains the compound of cube and octahedron. The model for $n = 4$ is shown above and below.



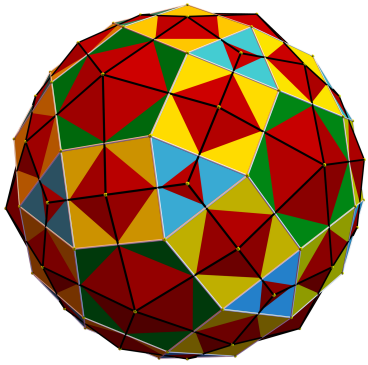
The slidables needed are shown below:



You will need

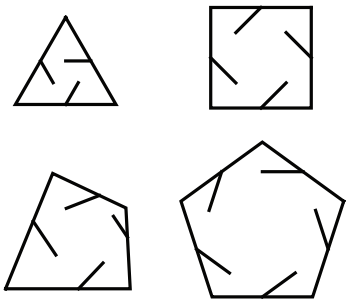
- 8 triangles
- 2 squares
- 8 deltoids

21 Rhombicosidodecahedron with Dual

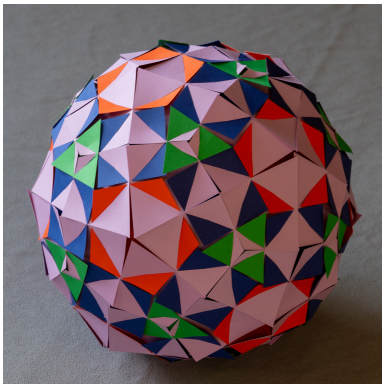


The Rhombicosidodecahedron is an Archimedean solid having triangles, squares, and pentagons as faces. Its dual is the Deltoidal Hexecontahedron, all of whose faces are deltoids. The compound consists of

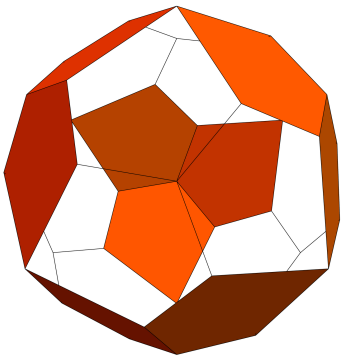
- 20 triangles
- 30 squares
- 12 pentagons
- 60 deltoids.

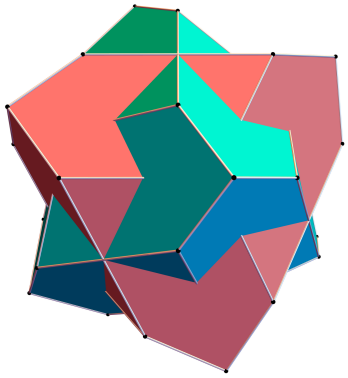


So you will need a total of 122 slidables to build this model, and, naturally, a lot of patience.



22 Two Tetartoids

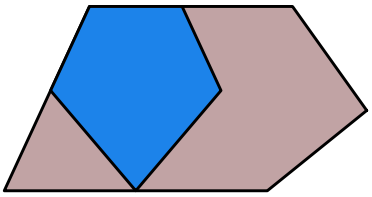




ROBERT AUSTIN writes in his blog **Robert Loves Pi** that 'the third stellation of the Pentagonal Icositetrahedron is a compound of two irregular dodecahedra'. His images are compelling, so I tried to remake his stellation. Unfortunately, he gives few clues, and on the website of the software he is using, a third stellation of the pentagonal icositetrahedron is mentioned, but it is different from his.

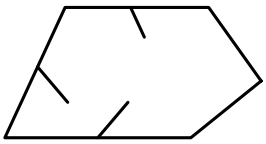
My starting point, therefore, was the Pentagonal Icositetrahedron. If Robert's claim were true, it should be possible to partition the 24 faces of the Pentagonal Icositetrahedron into two sets of 12 in a symmetric way. This is indeed the case; each set consists of four clusters of three pentagons:

The next step was to find a stellation of one of the two sets that results in a dodecahedron. Here is what becomes of a single pentoid:



The rest was rather mechanical. Below you see a projection of the compound.

To build the model, you will need 24 irregular pentagons.



The irregularity of the pentagons makes the assembly quite a puzzle.

The two irregular dodecahedra in this compound are special instances of tetartoids. A tetartoid is a dodecahedron with a tetrahedral symmetry and all congruent faces. The tetartoids come as a 3-parameter family.

The tetrahedral symmetry of the two tetardoids makes this compound look like an elaborate version of the Stella Octangula, where

three elongated pentagons have replaced each triangular face.

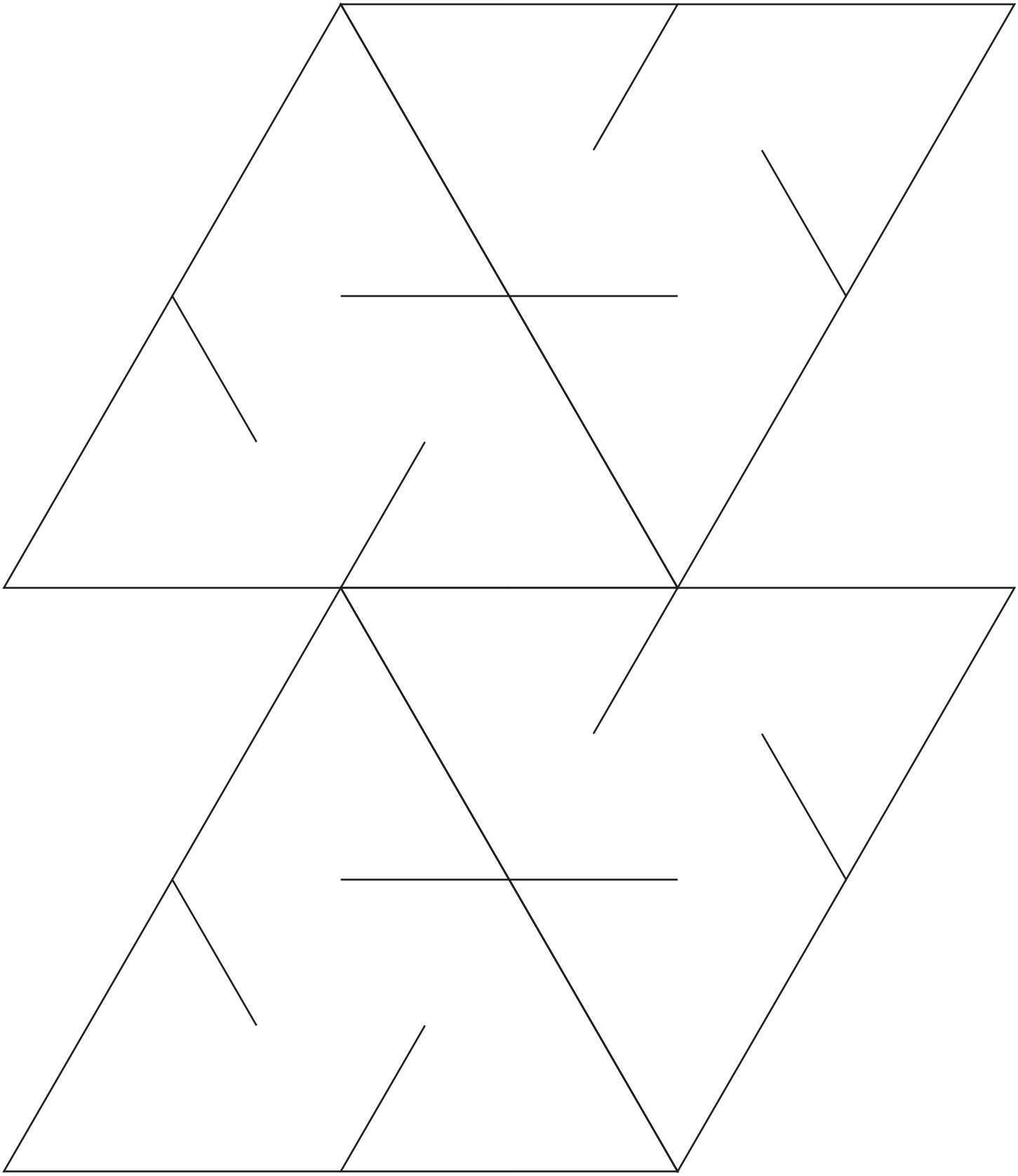
It is hard to convey how this compound looks in images; one has to hold it in one's hands.

23 Templates

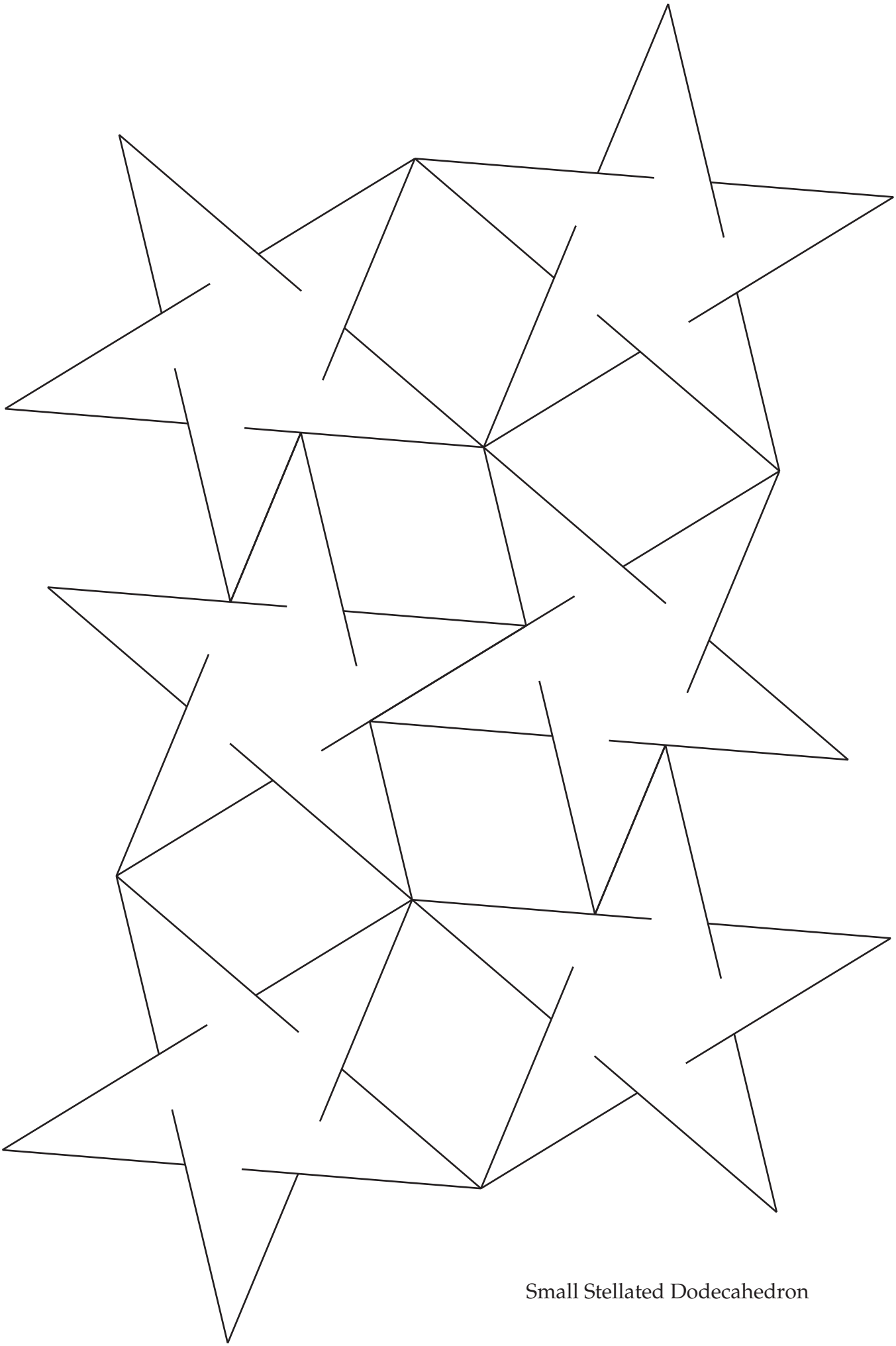
The following pages contain templates for making slides of all shapes on letter-sized paper. One way to use them is to print them on card stock and use a craft knife to cut out the slidables, cutting also along all slits. Some shapes need multiple copies of single pages, so always consult the description for how many slidables you need. If you want to make larger or smaller models, scale all pages by the same scale factor so that corresponding slits keep having the same length. There are also various die cutters available that will take over the cutting for you. The PDFs contain vector graphics, and it is likely possible to extract them and import them into the software that controls the die cutter.

Personally, I have been using a Silhouette die cutter as follows: I prepare the templates in Adobe Illustrator, export them to DXF (using the 2013 AutoCAD format), and import them into the free version of the Silhouette software. As I import the DXF graphics, I add a letter-sized rectangular frame in Illustrator. I then manually scale the imported DXF so that the frame fills the template, and then delete the frame. This allows me to scale various imported pages consistently.

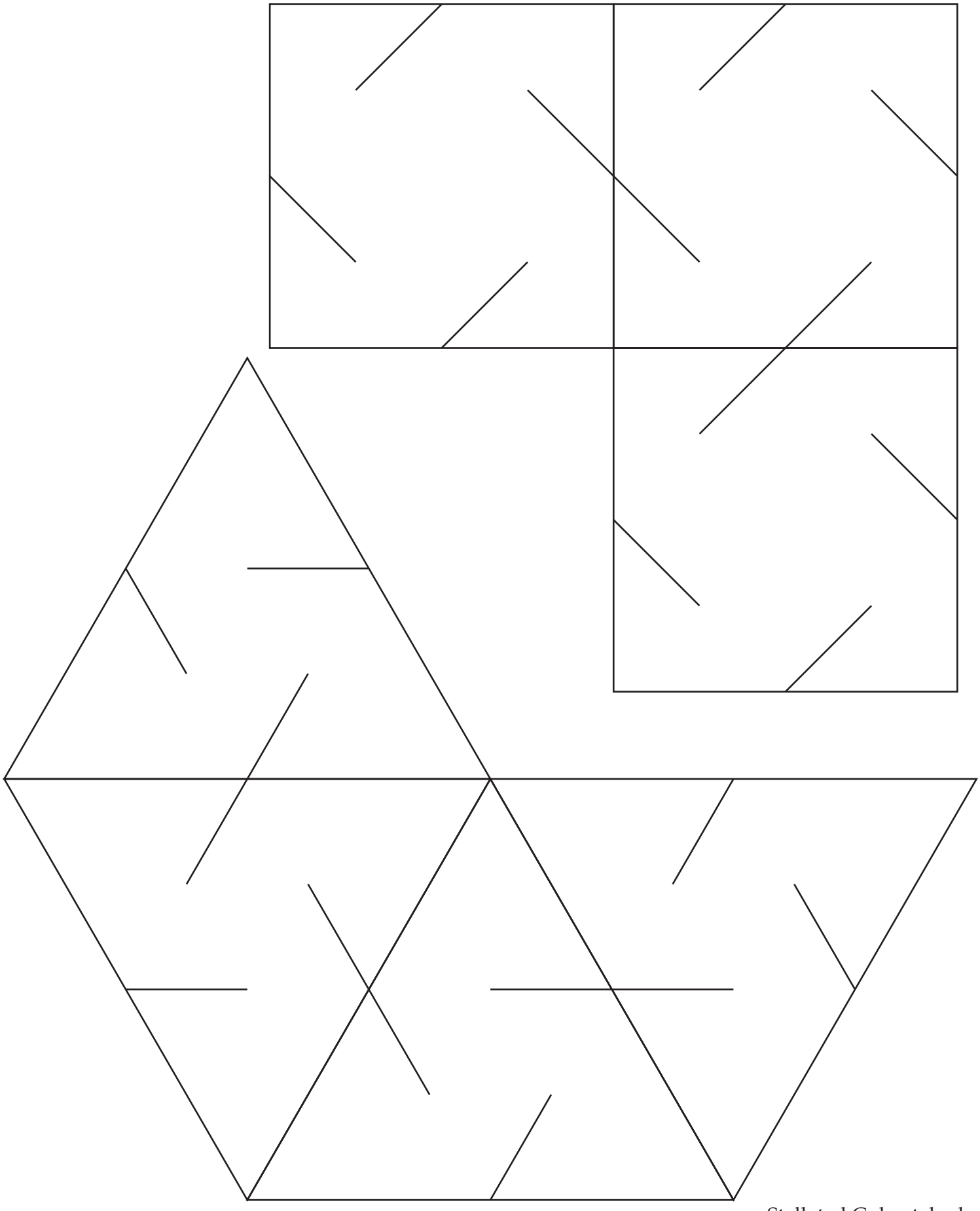
The competition (Cricut) apparently allows importing SVG files, so a similar approach should work, but I haven't tried it as I don't own one of their machines.



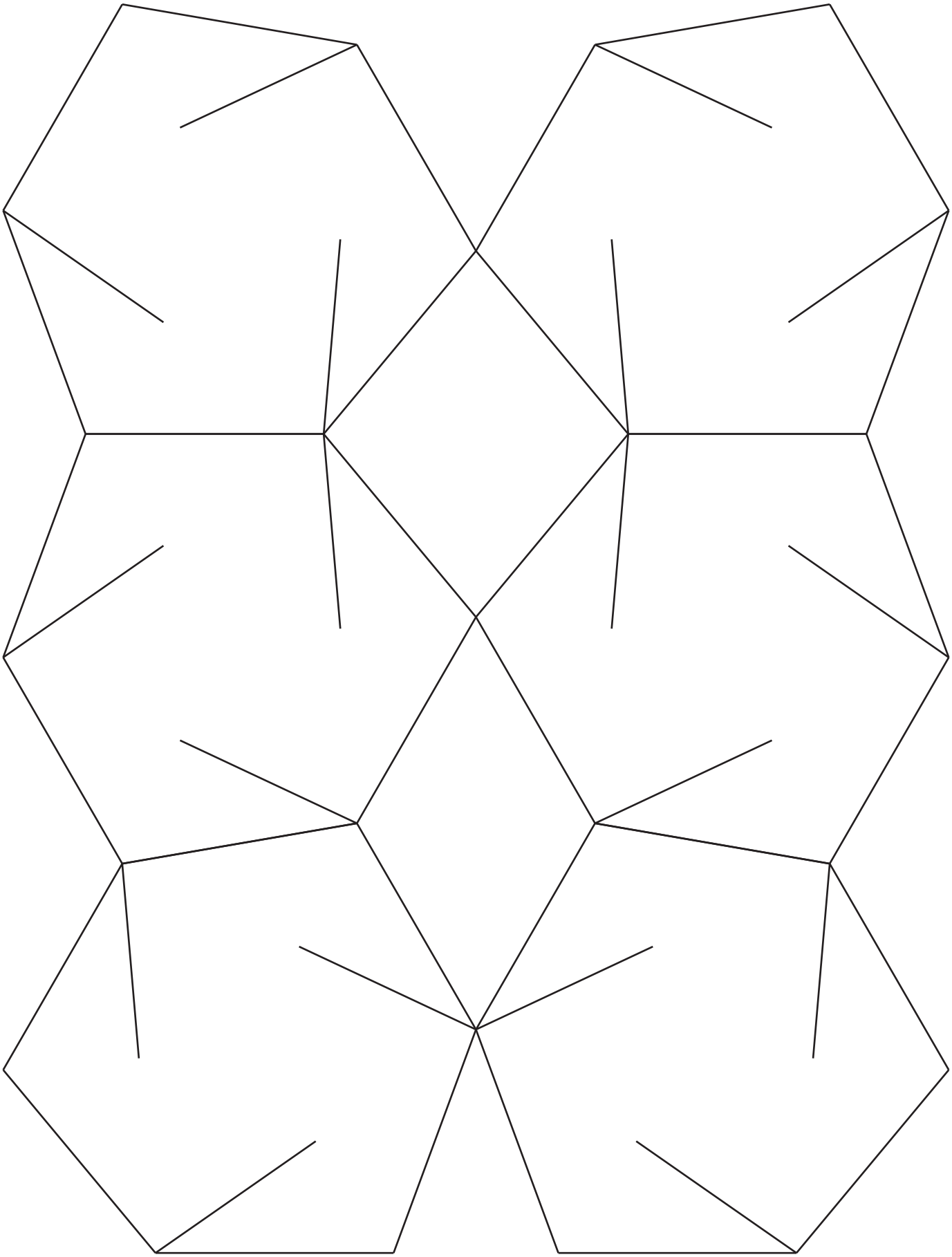
Stella Octangula



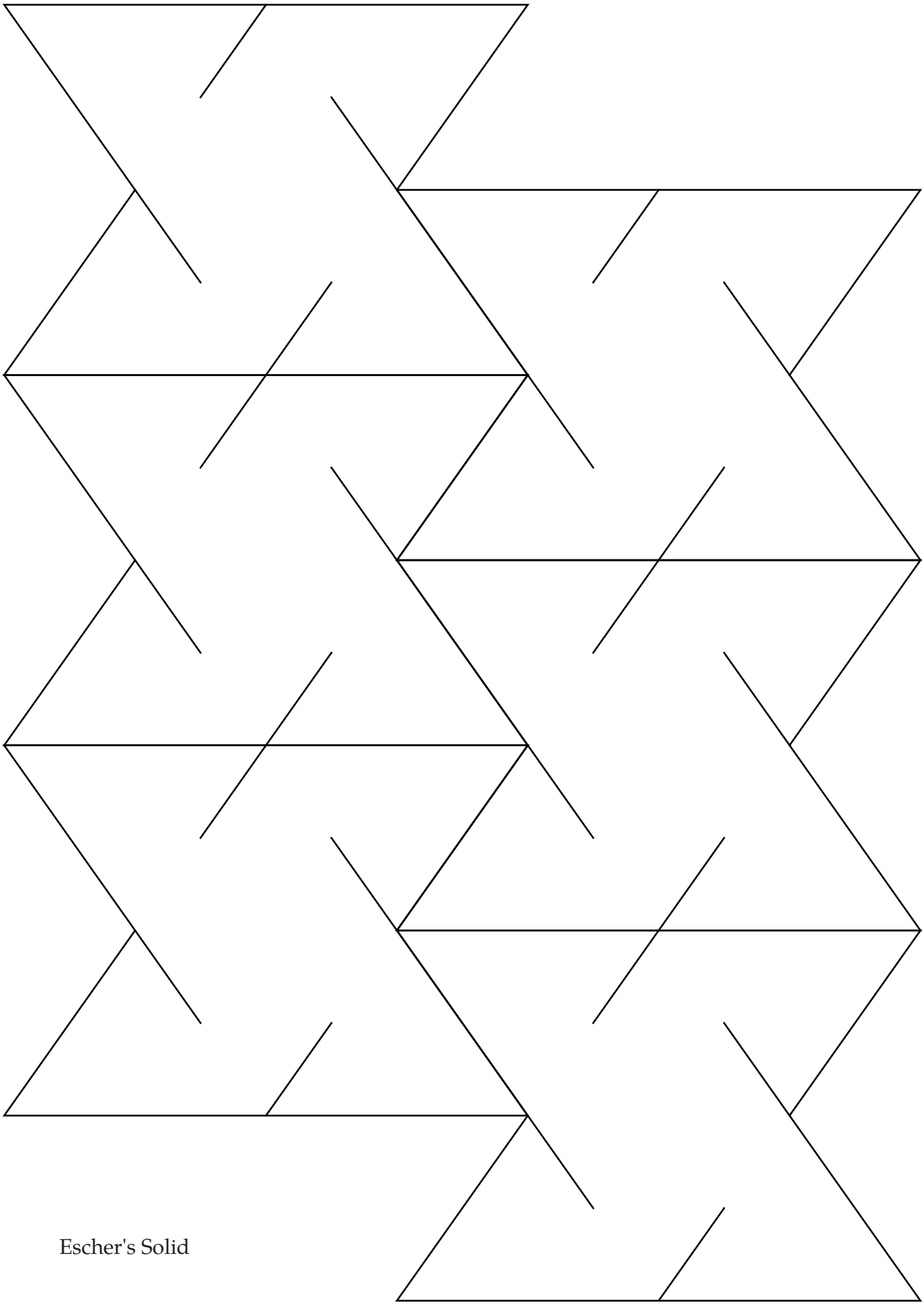
Small Stellated Dodecahedron



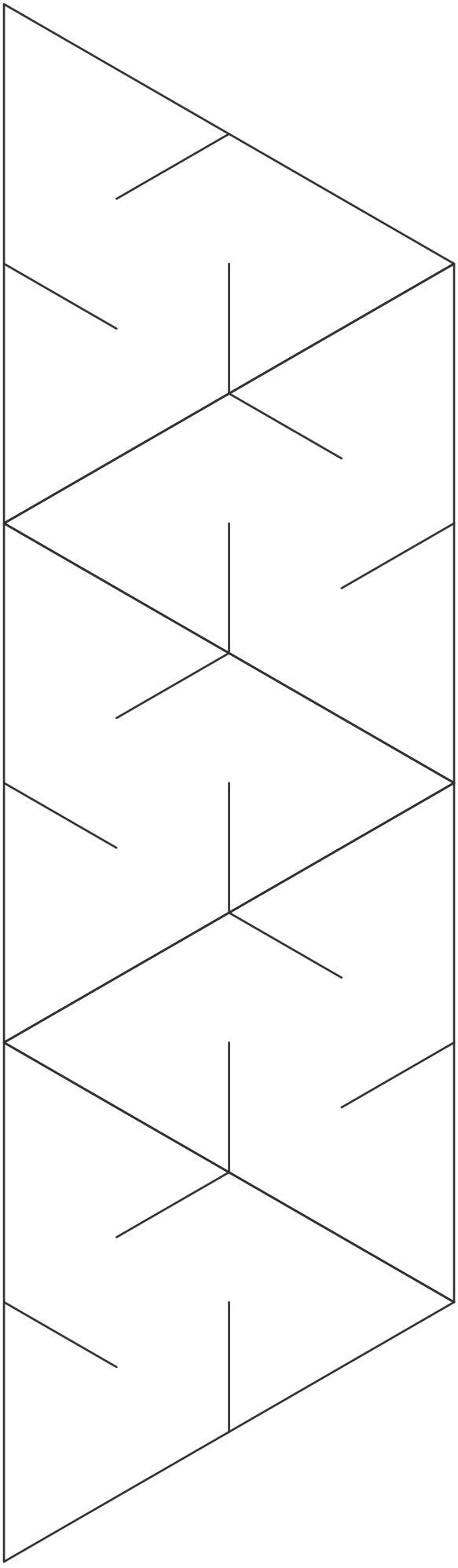
Stellated Cuboctahedron

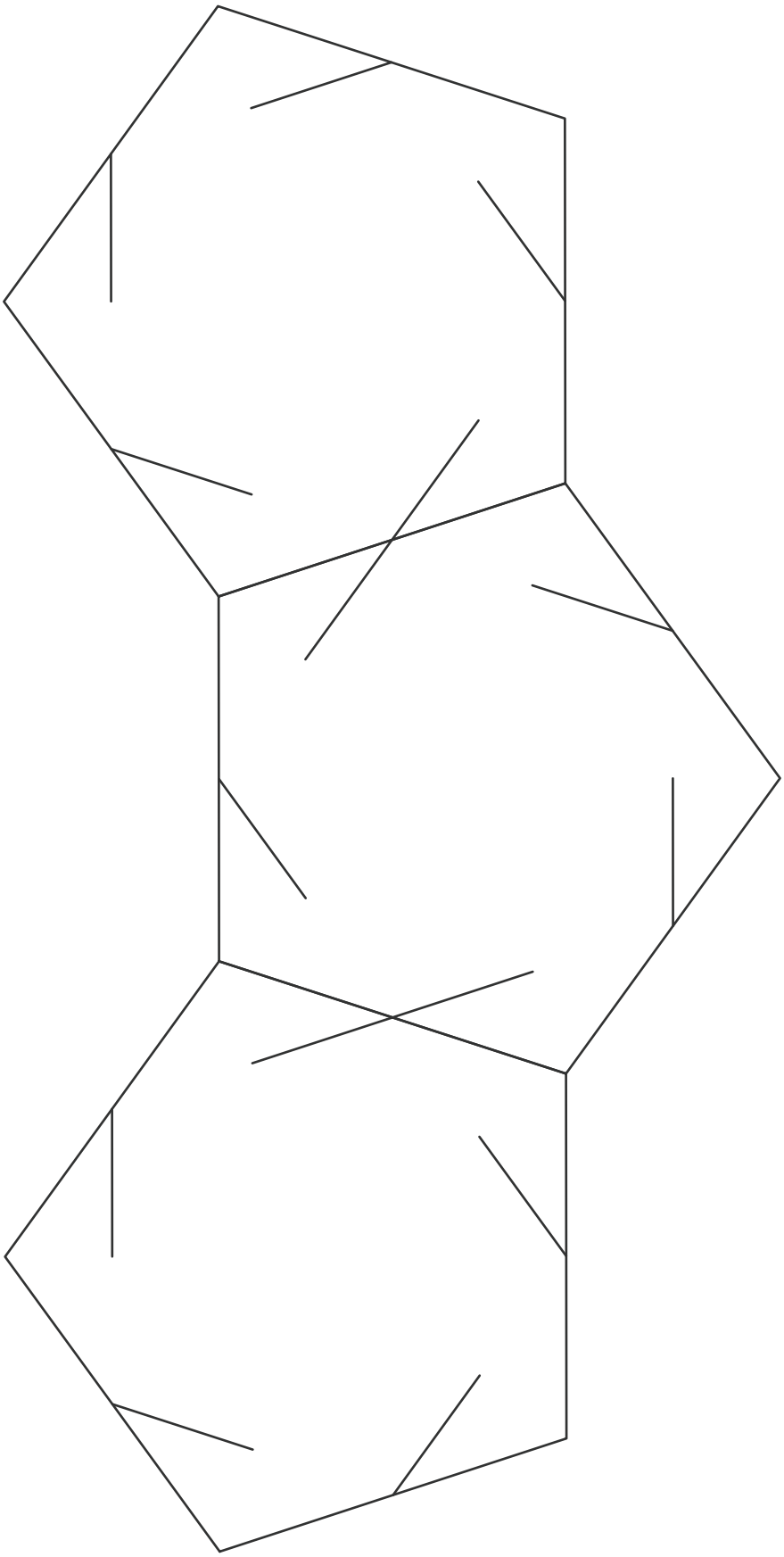


Small Triambic Icosahedron

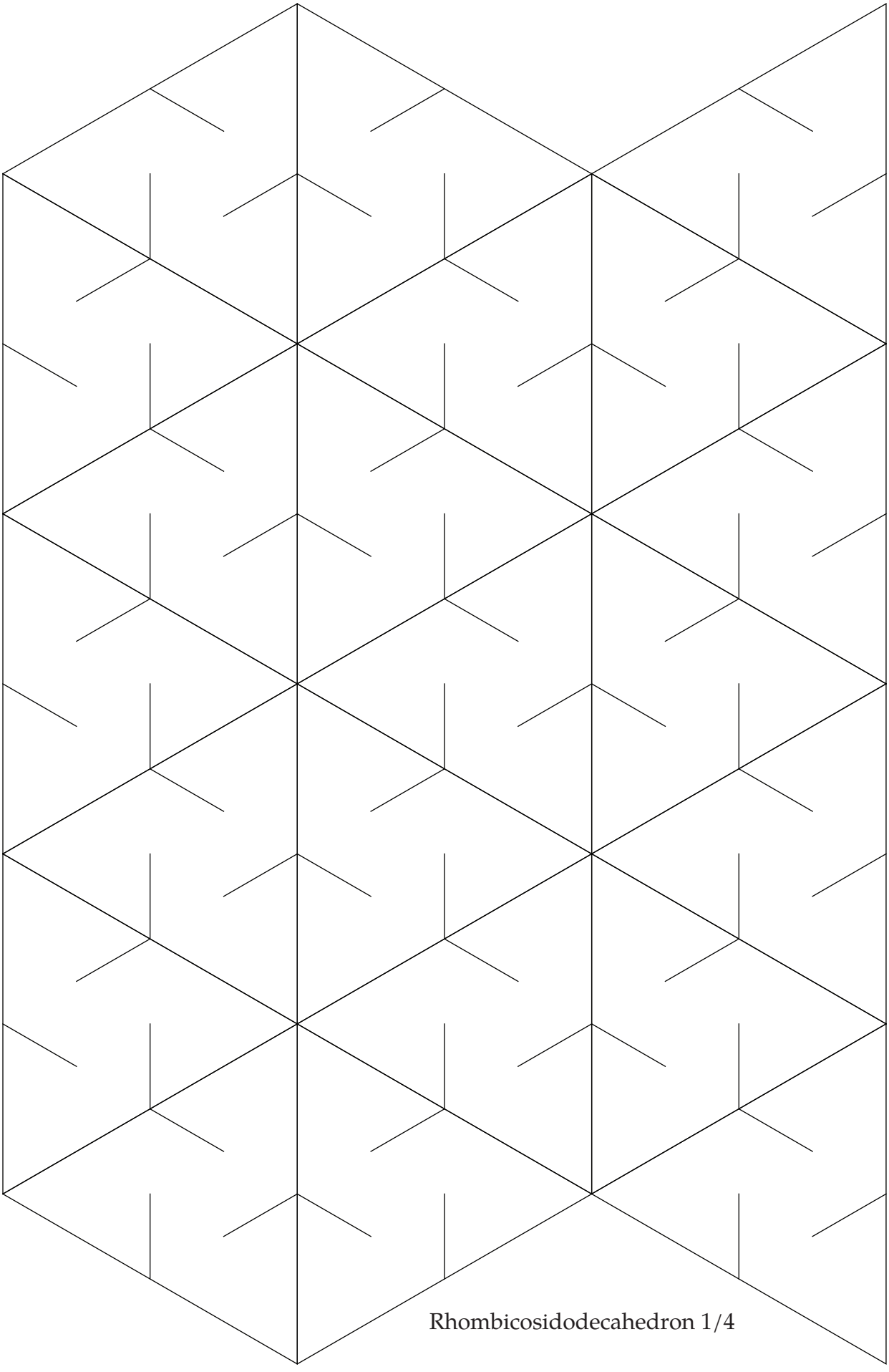


Escher's Solid

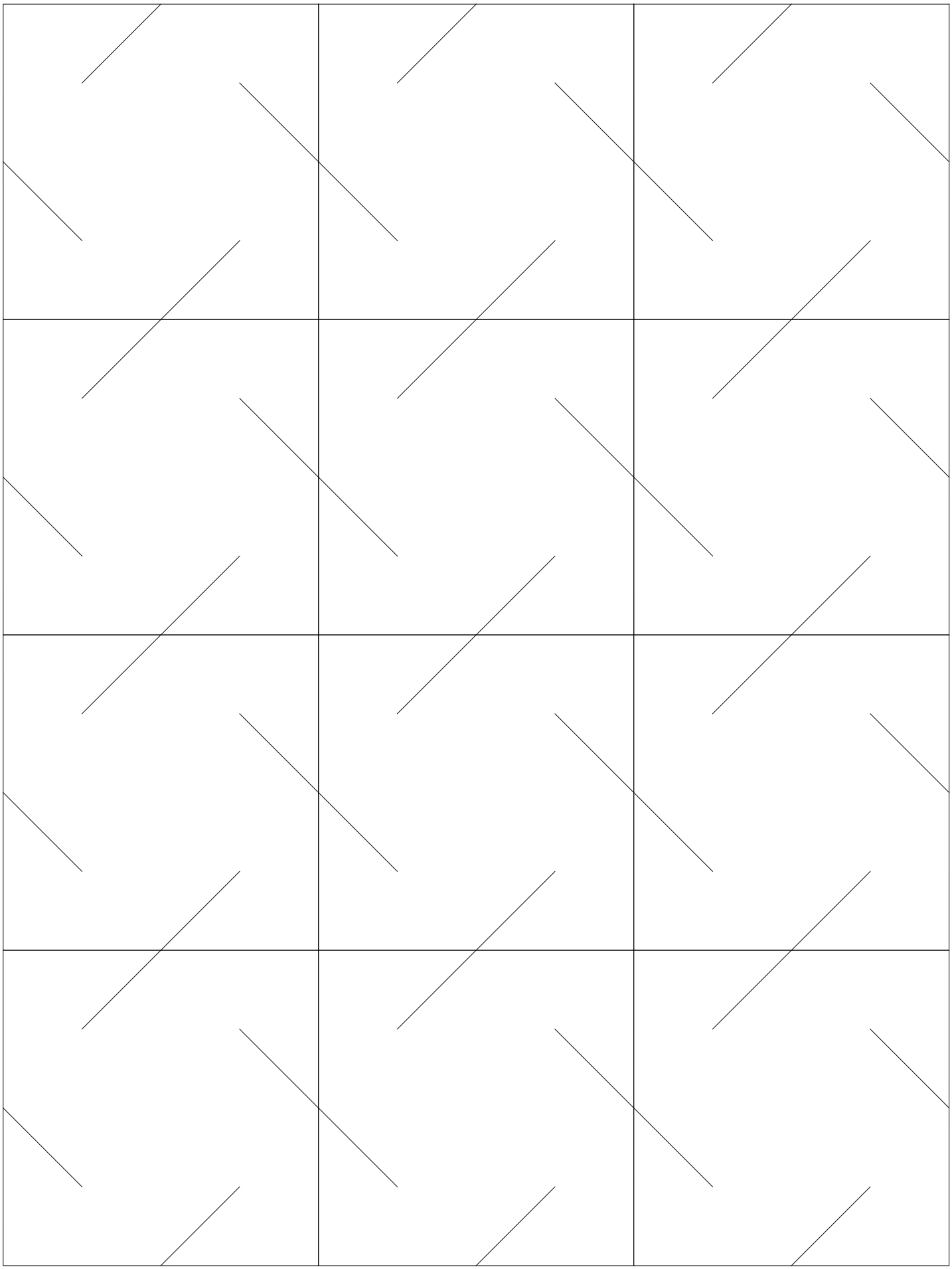




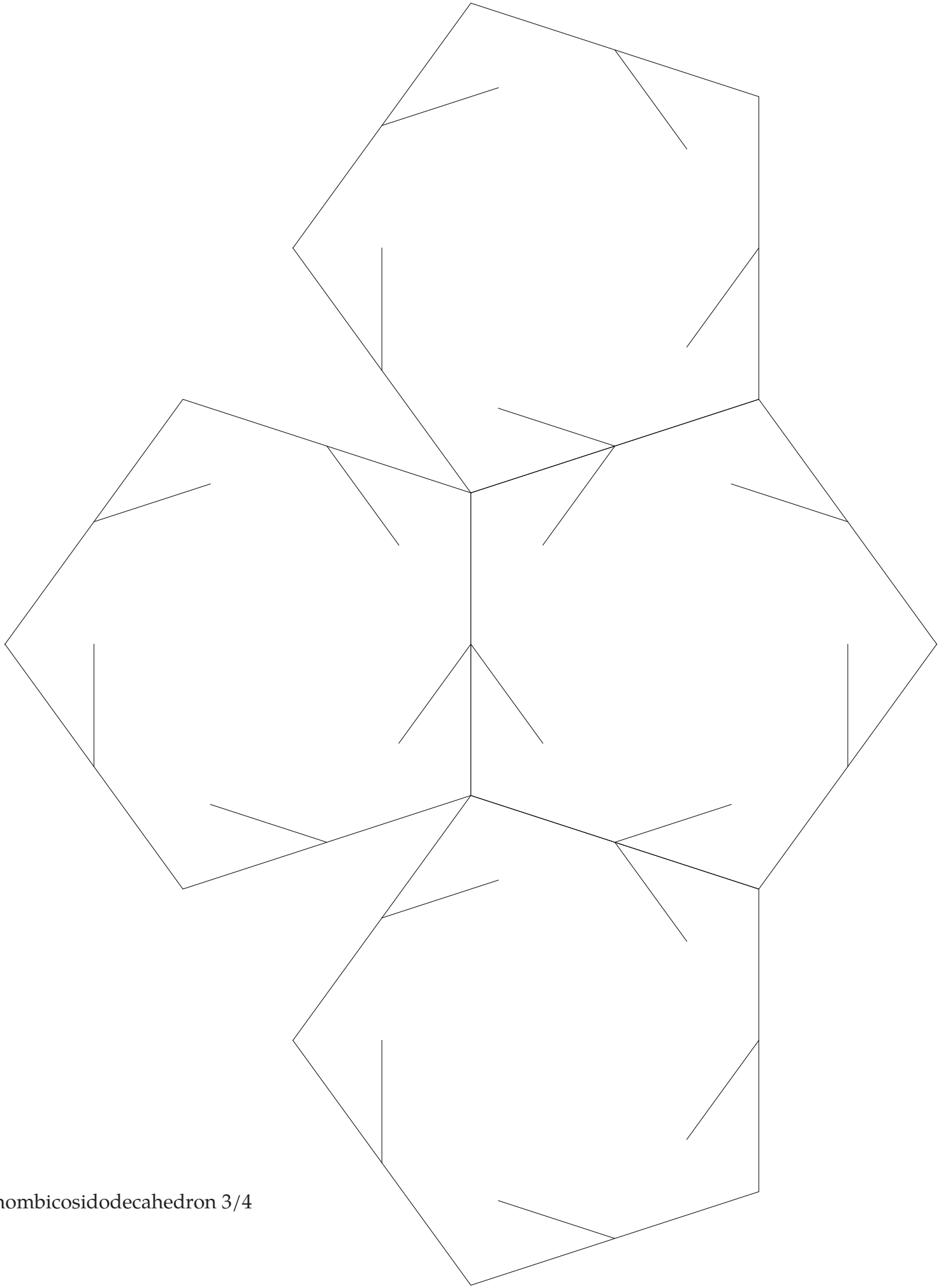
Stellated DodecaIcosahedron



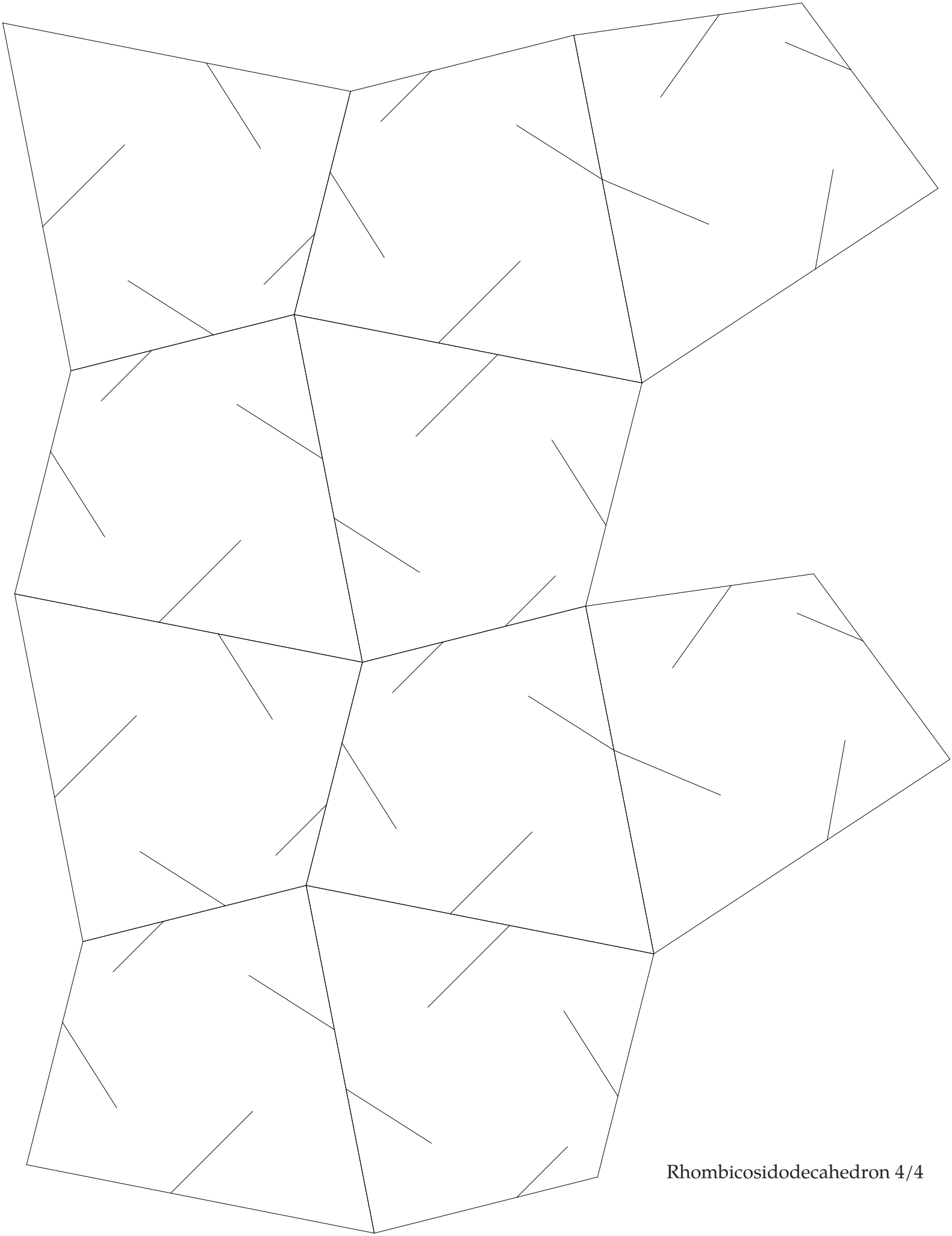
Rhombicosidodecahedron 1/4



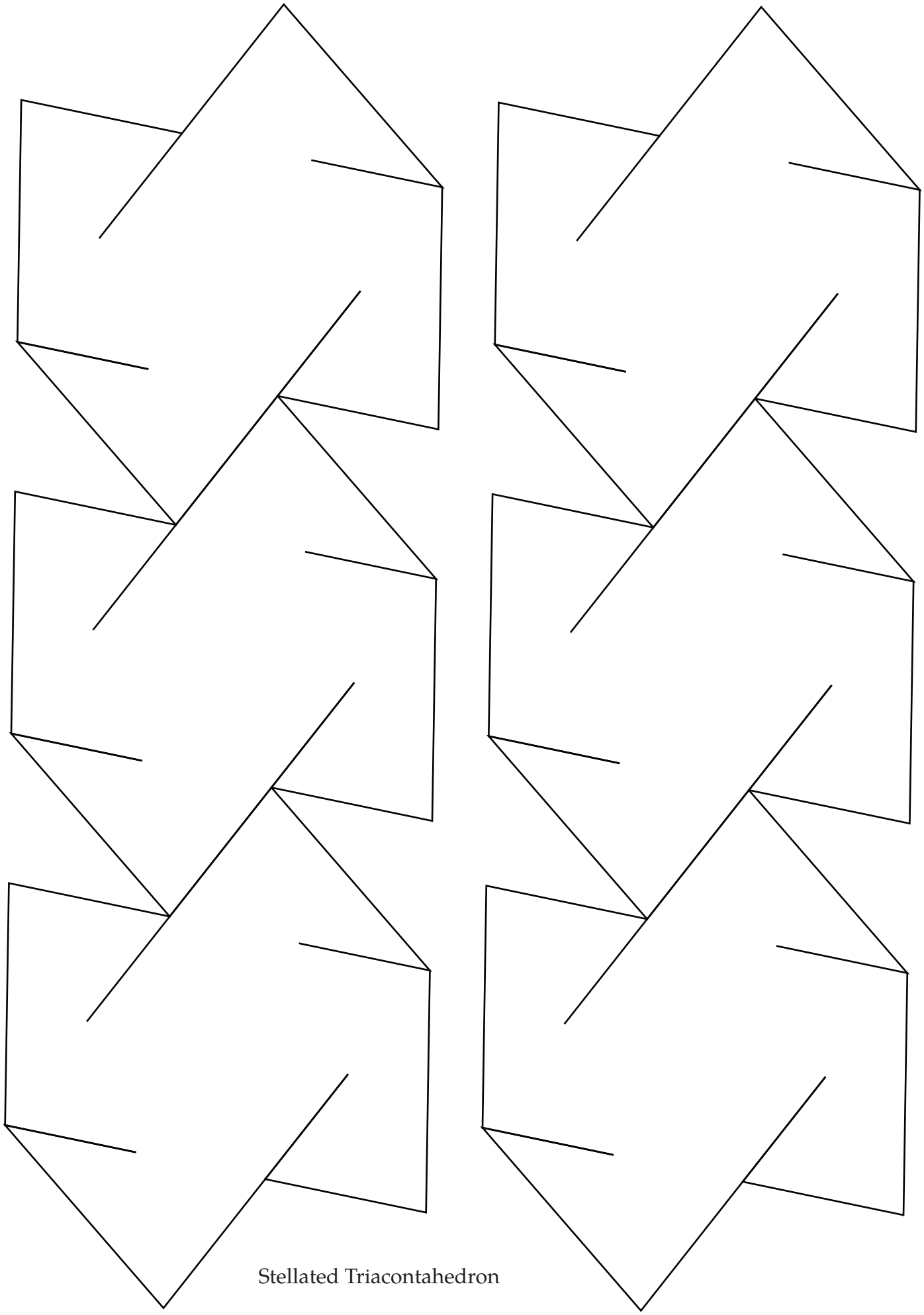
Rhombicosidodecahedron 2/4



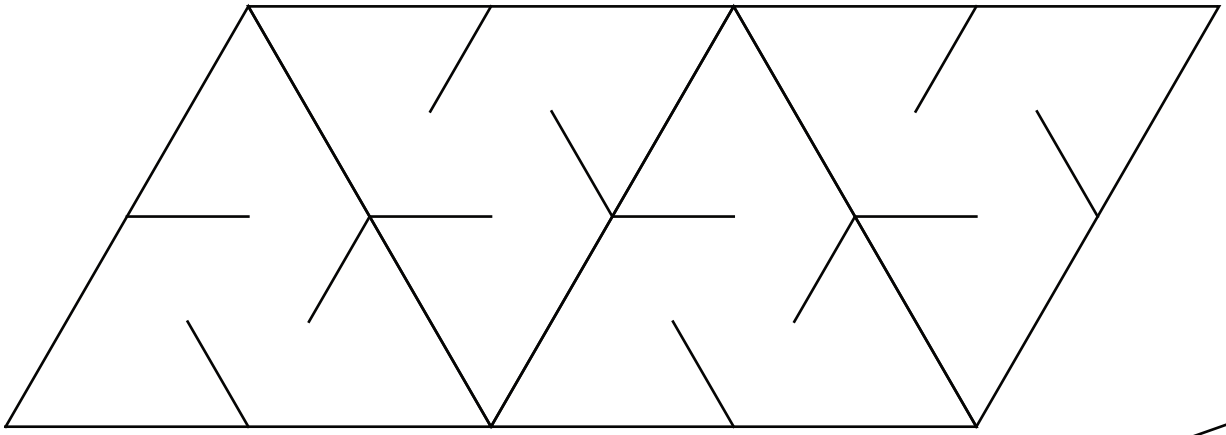
Rhombicosidodecahedron 3/4

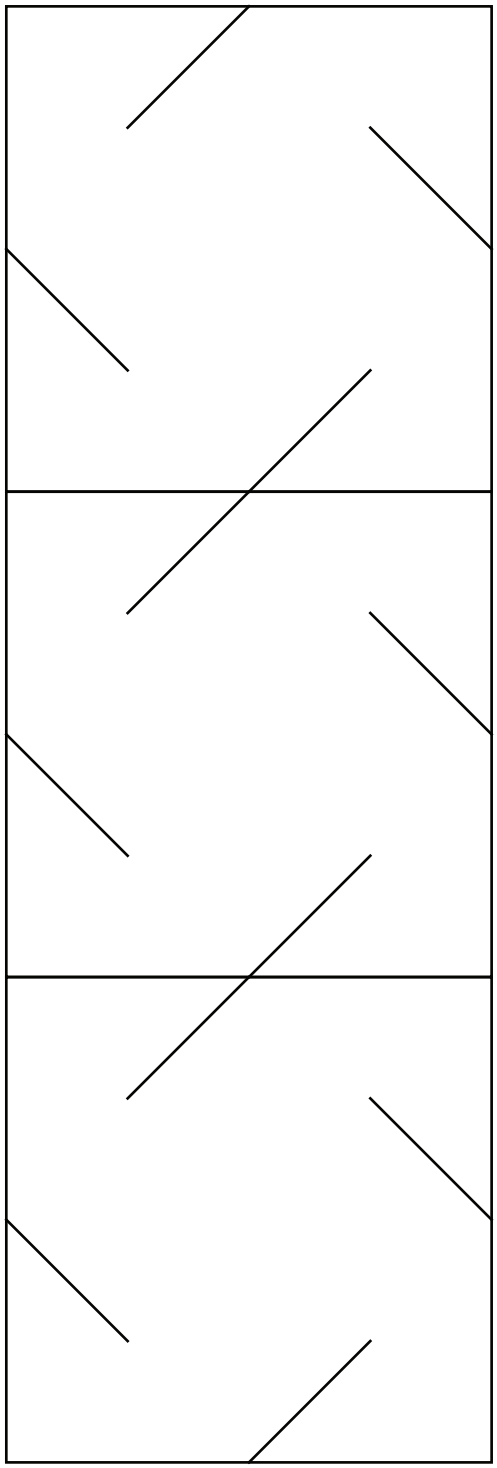


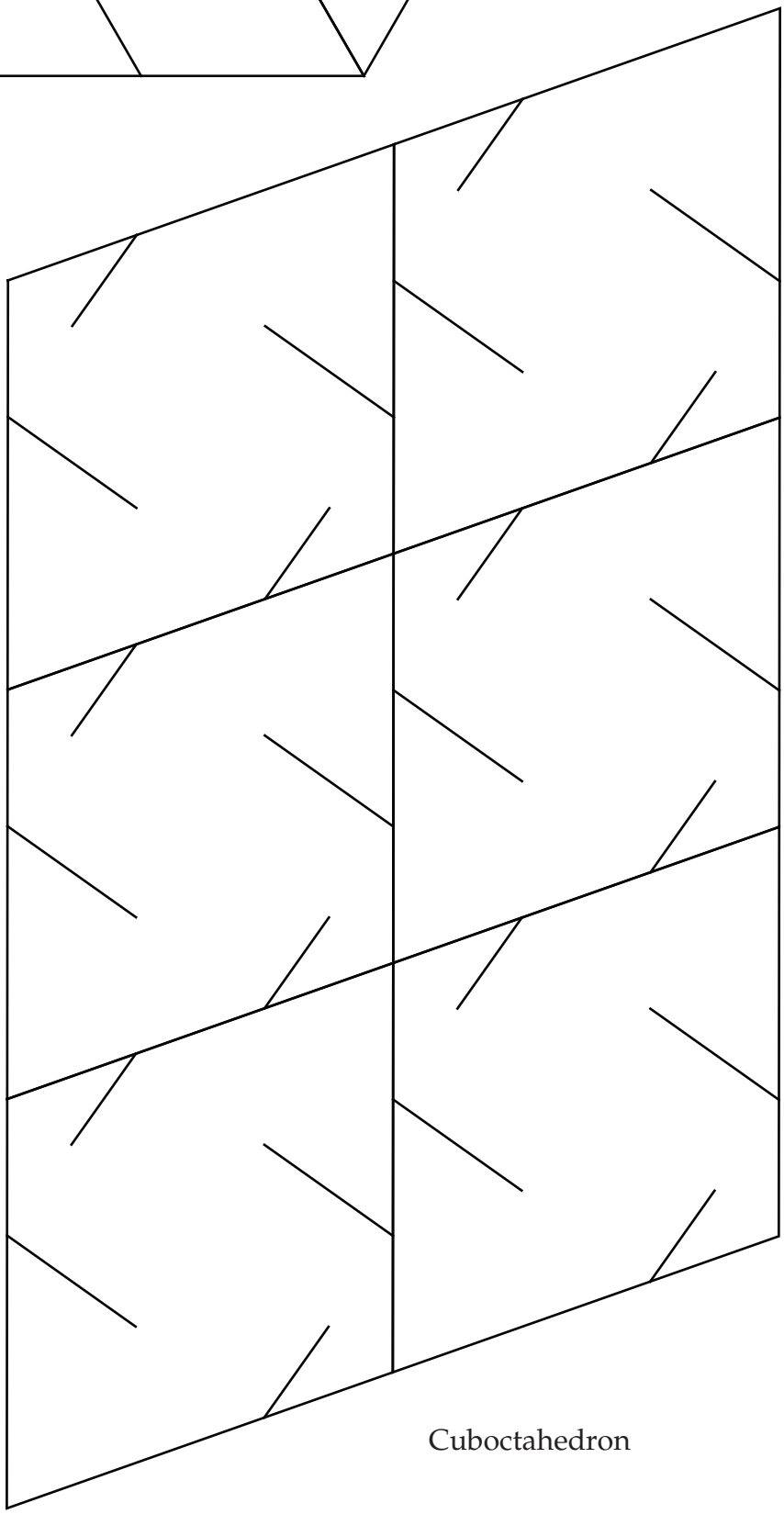
Rhombicosidodecahedron 4/4



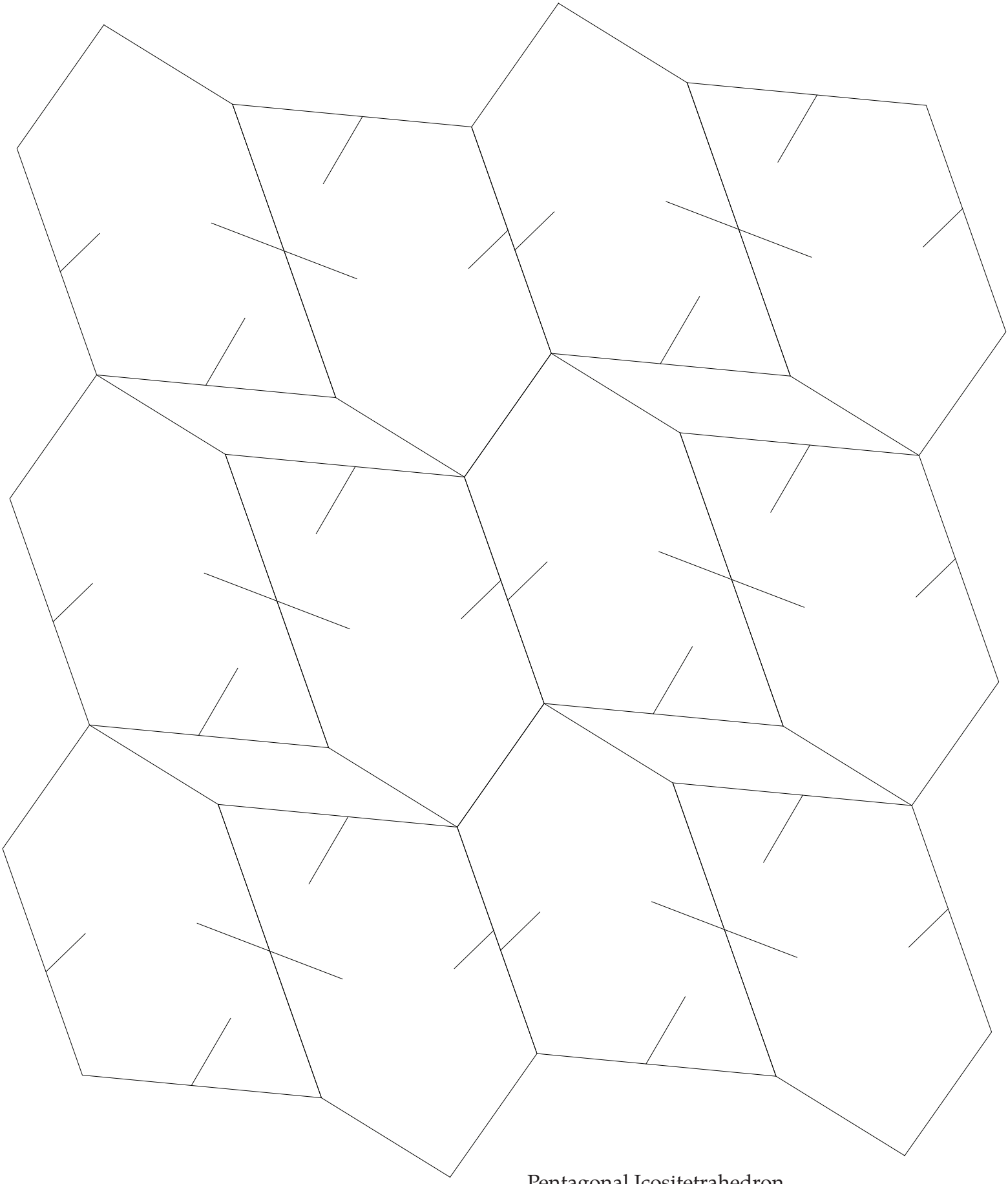
Stellated Triacontahedron



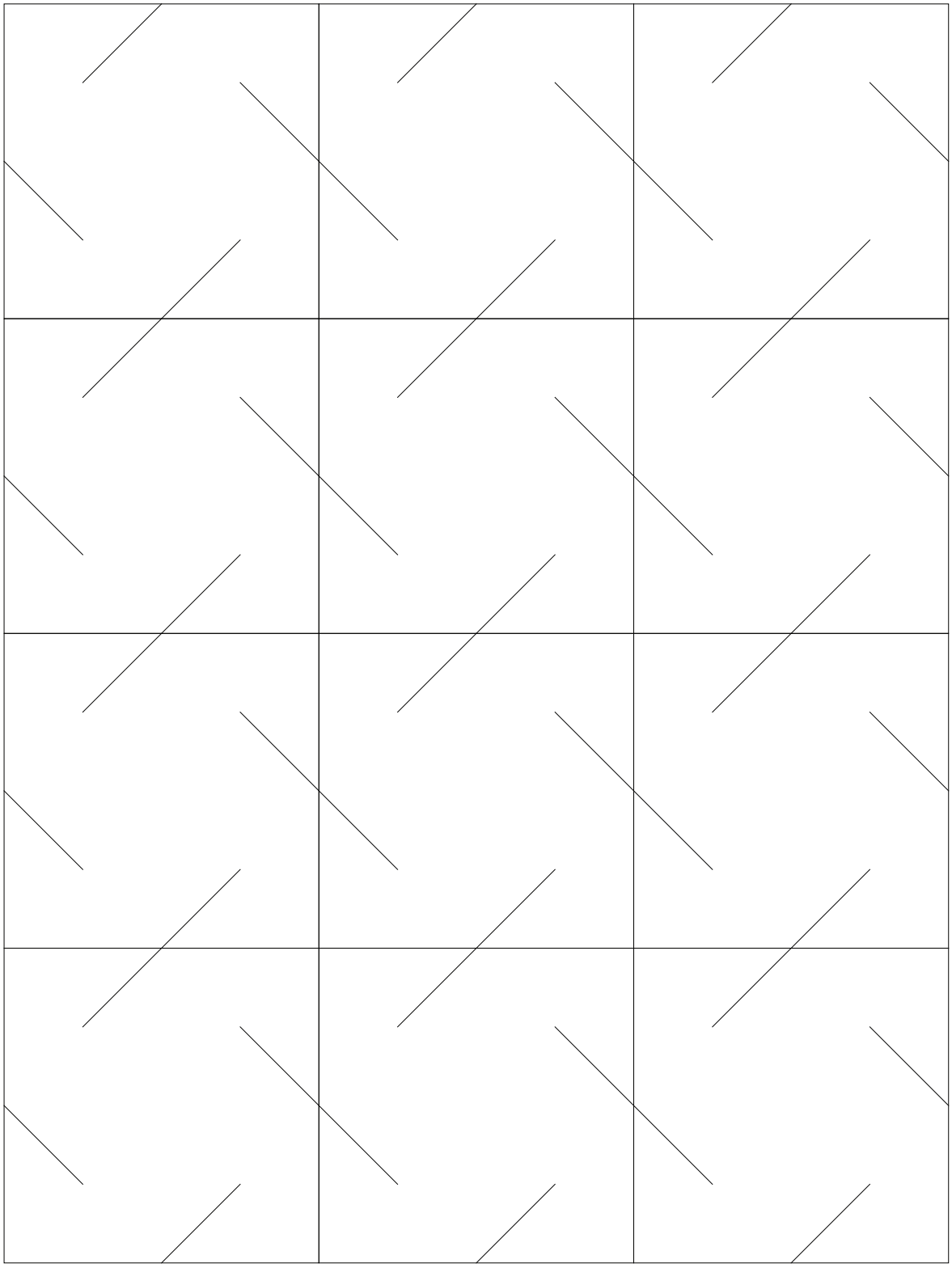




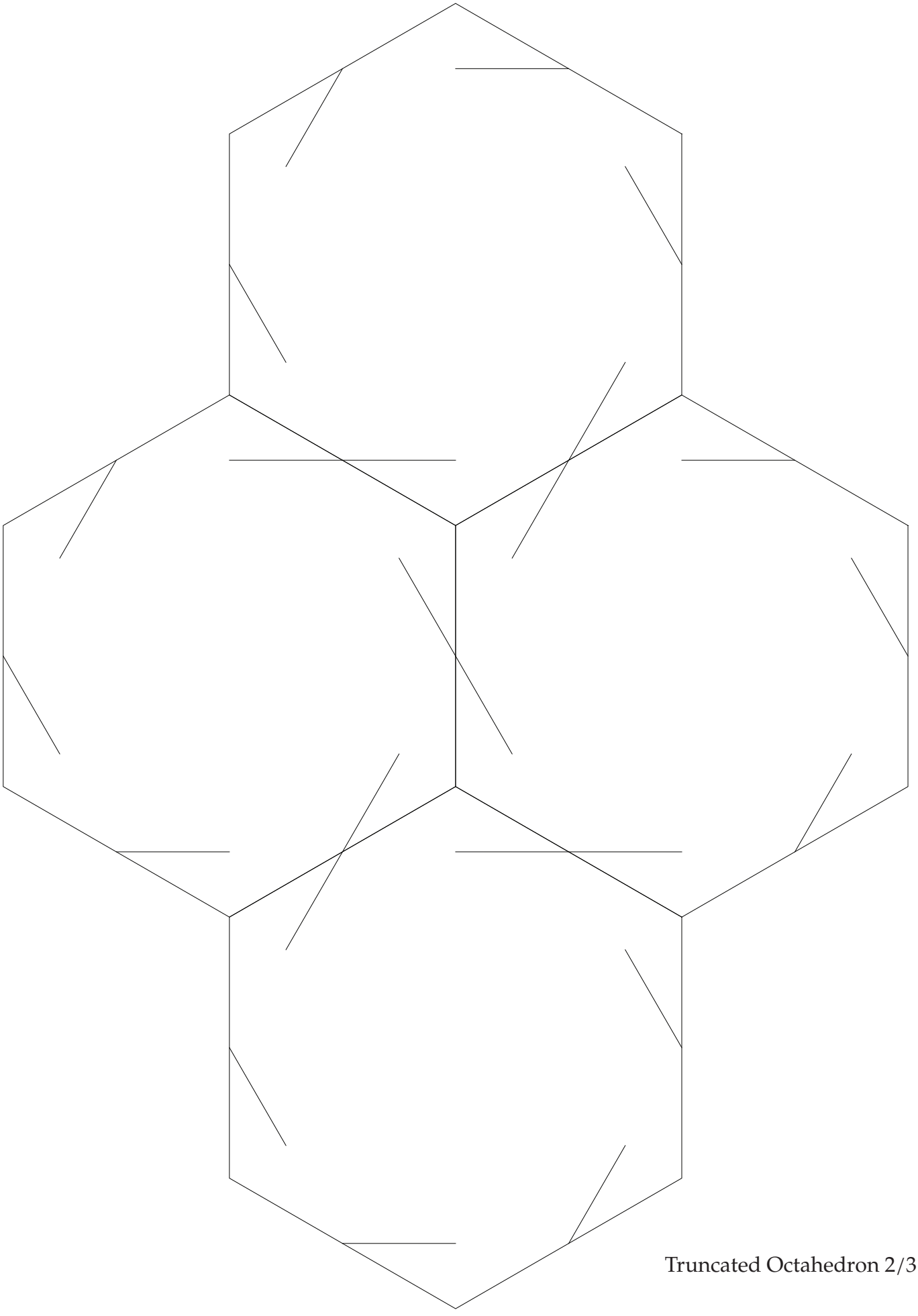
Cuboctahedron



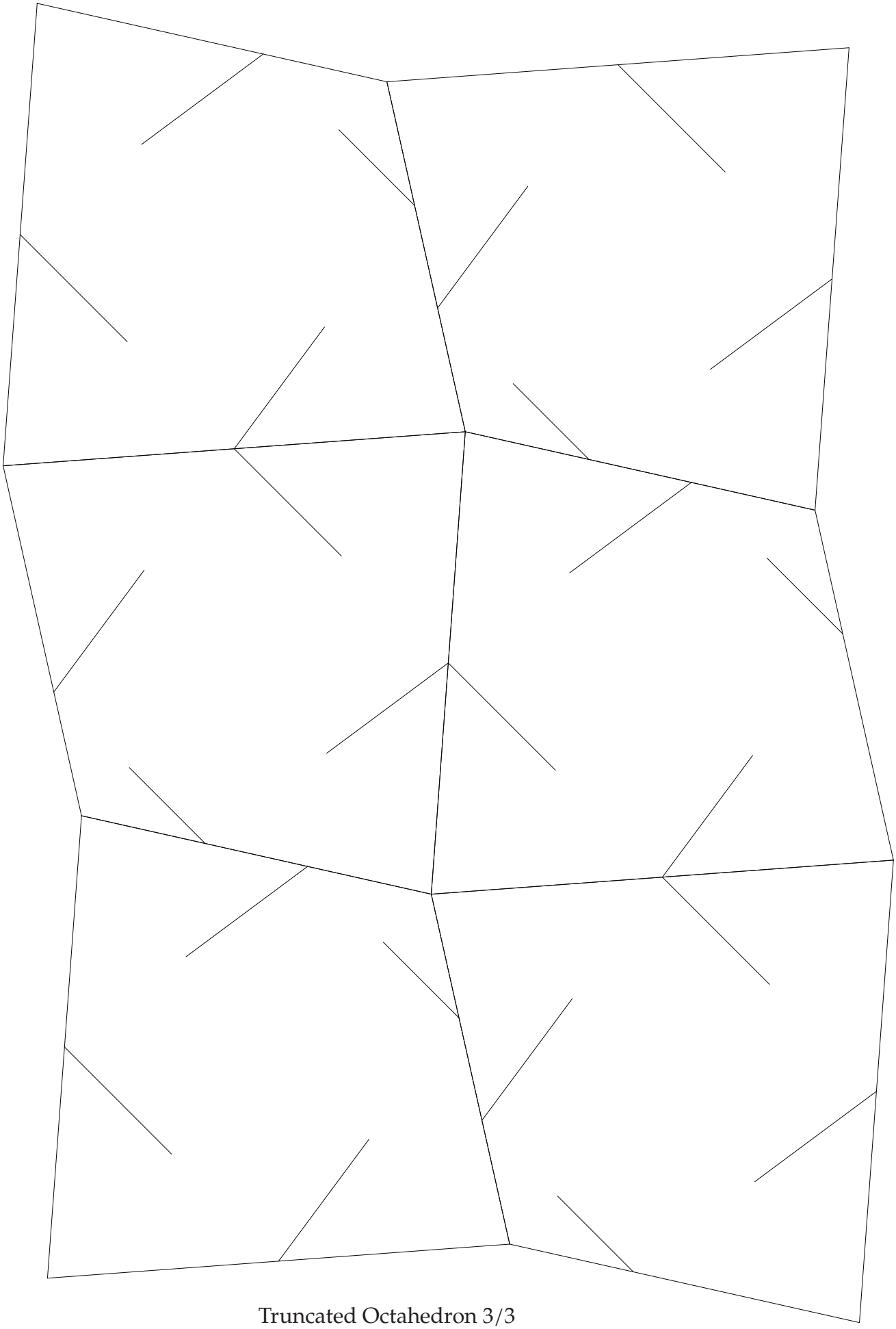
Pentagonal Icositetrahedron



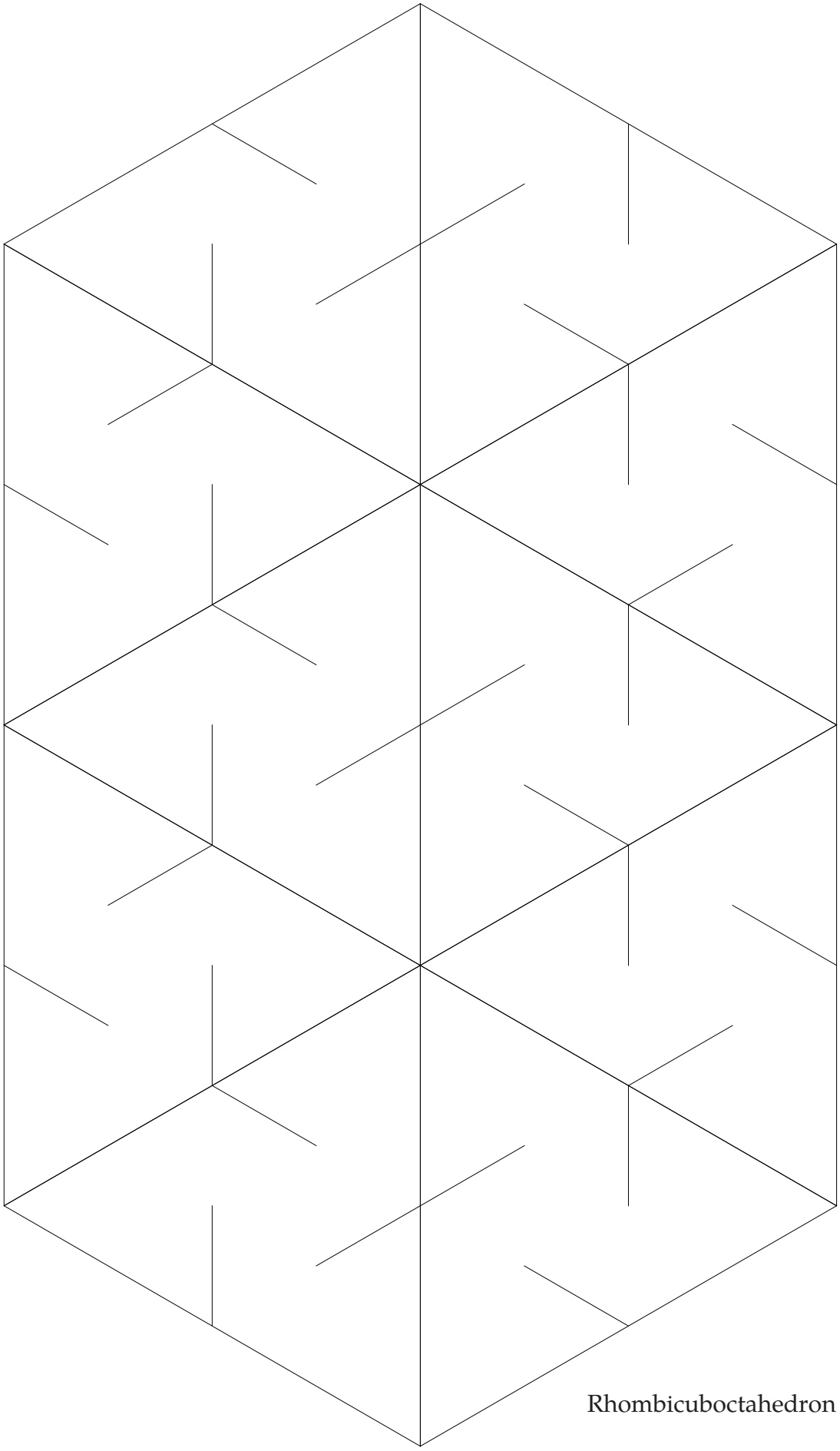
Truncated Octahedron 1/3



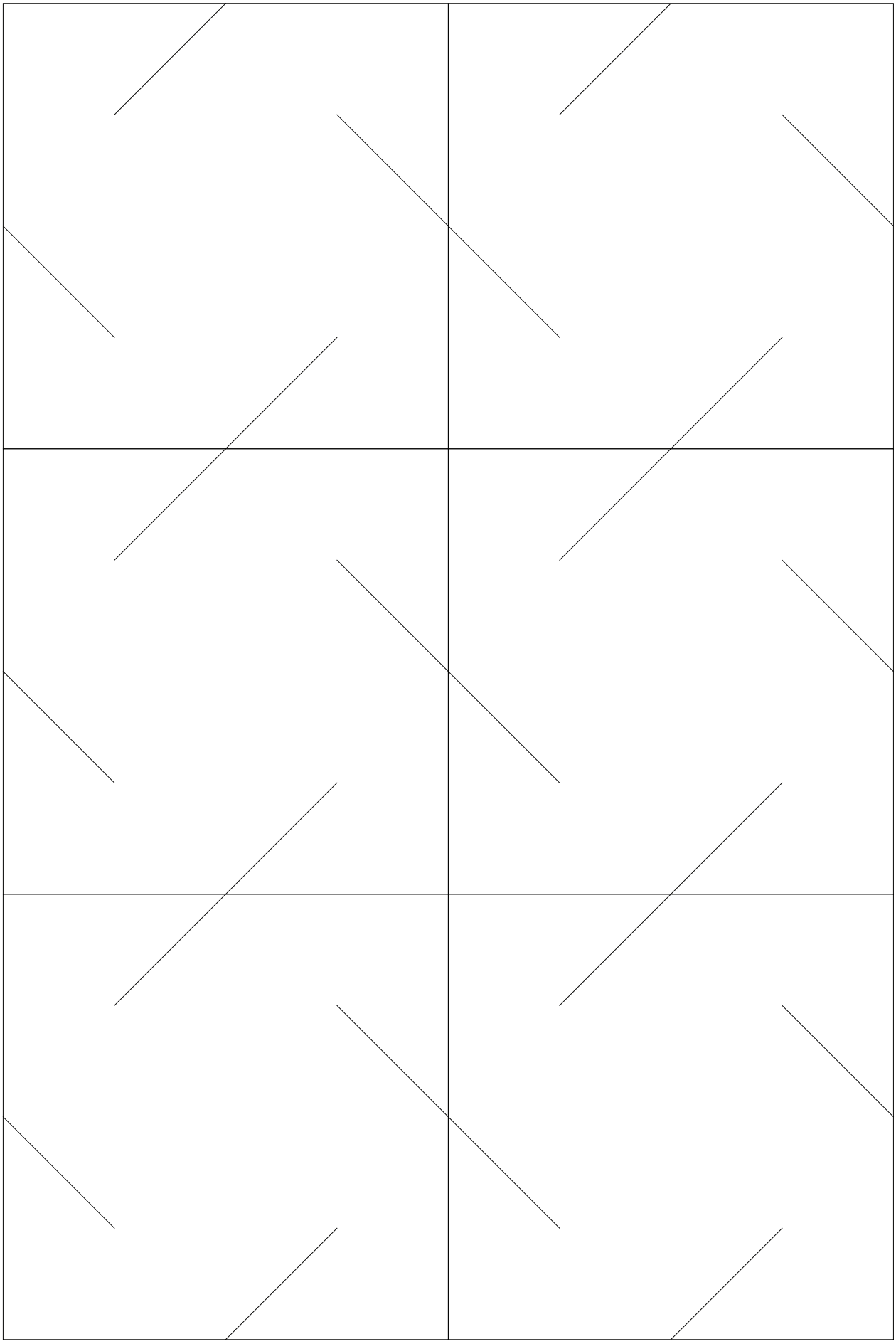
Truncated Octahedron 2/3

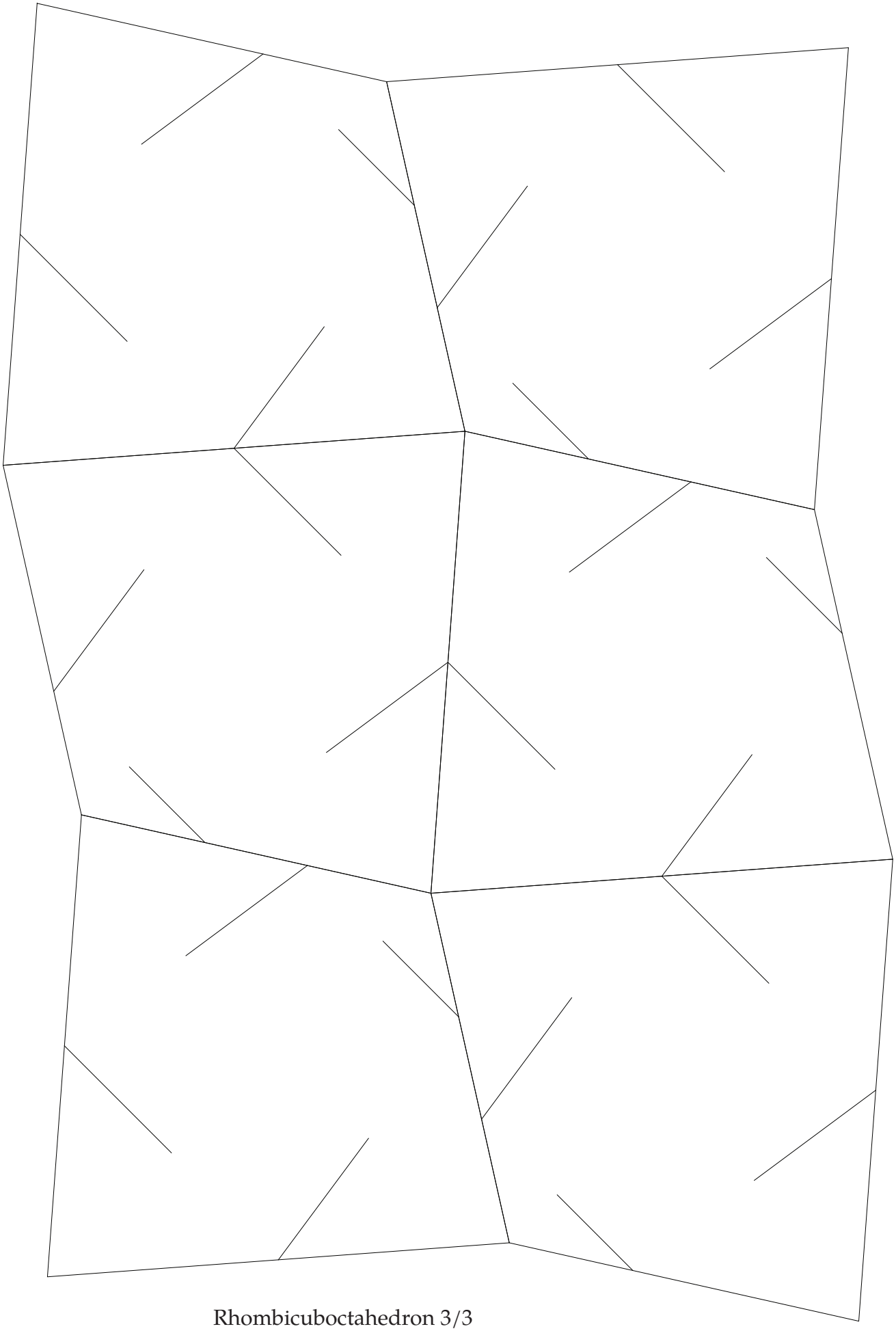


Truncated Octahedron 3/3

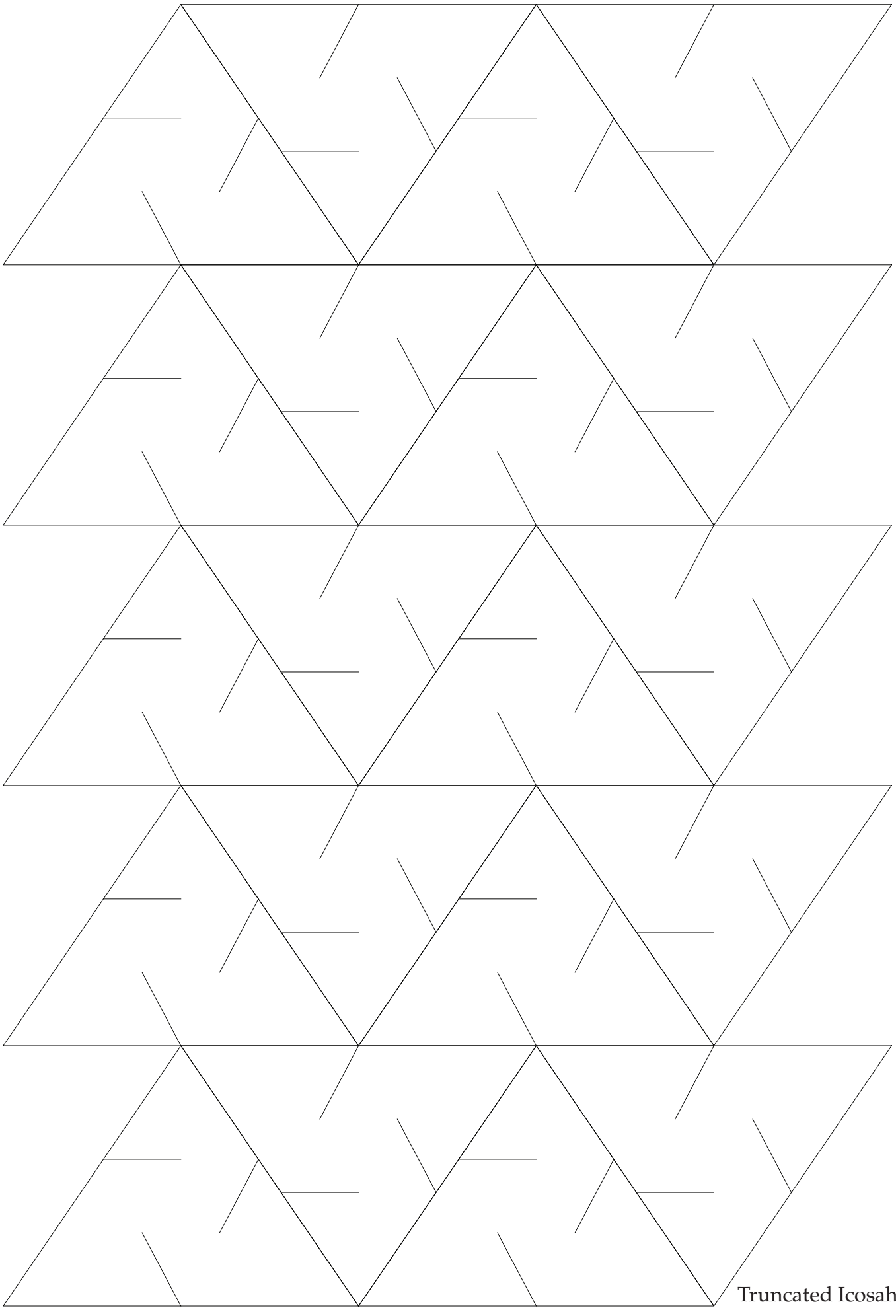


Rhombicuboctahedron 1/3

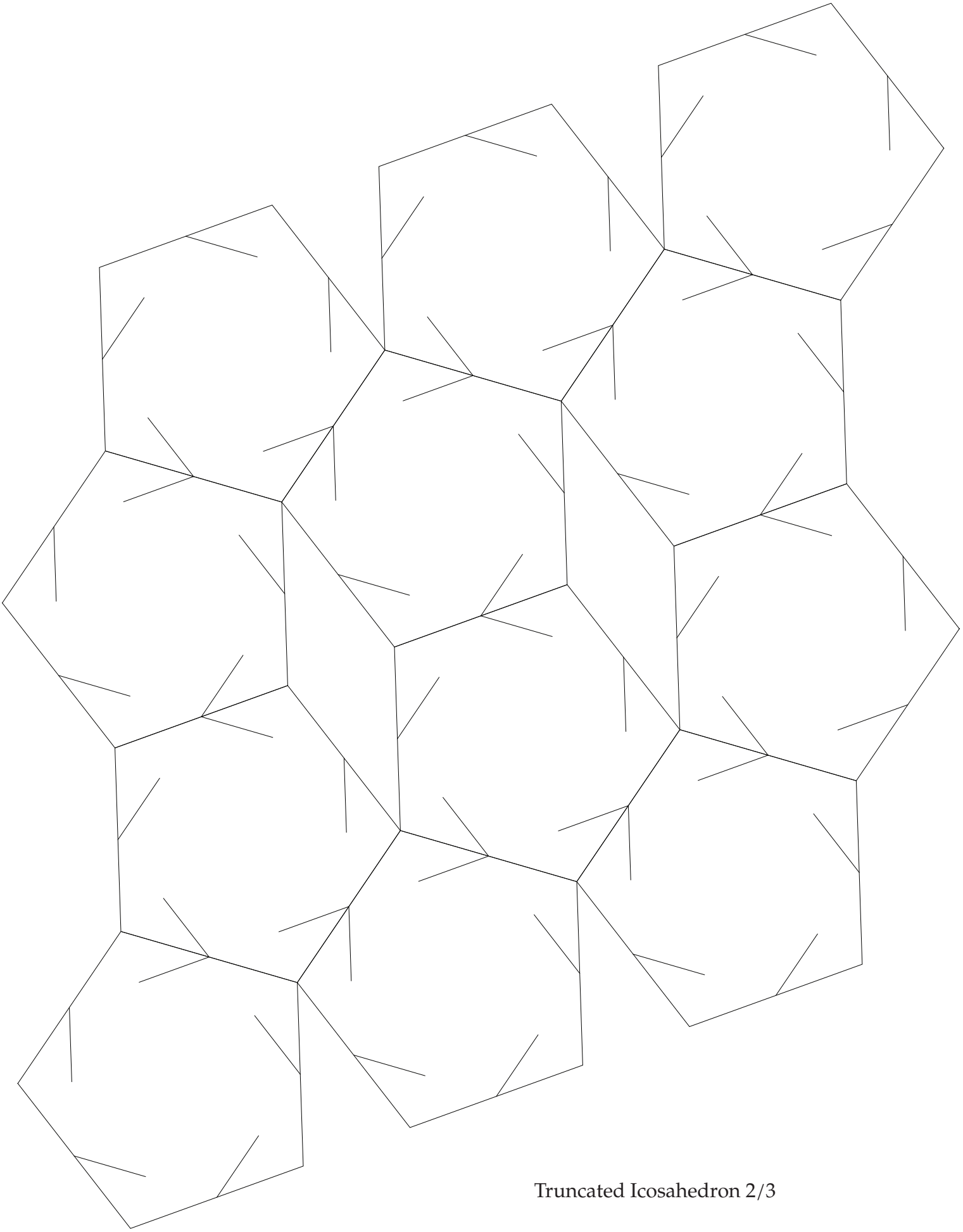




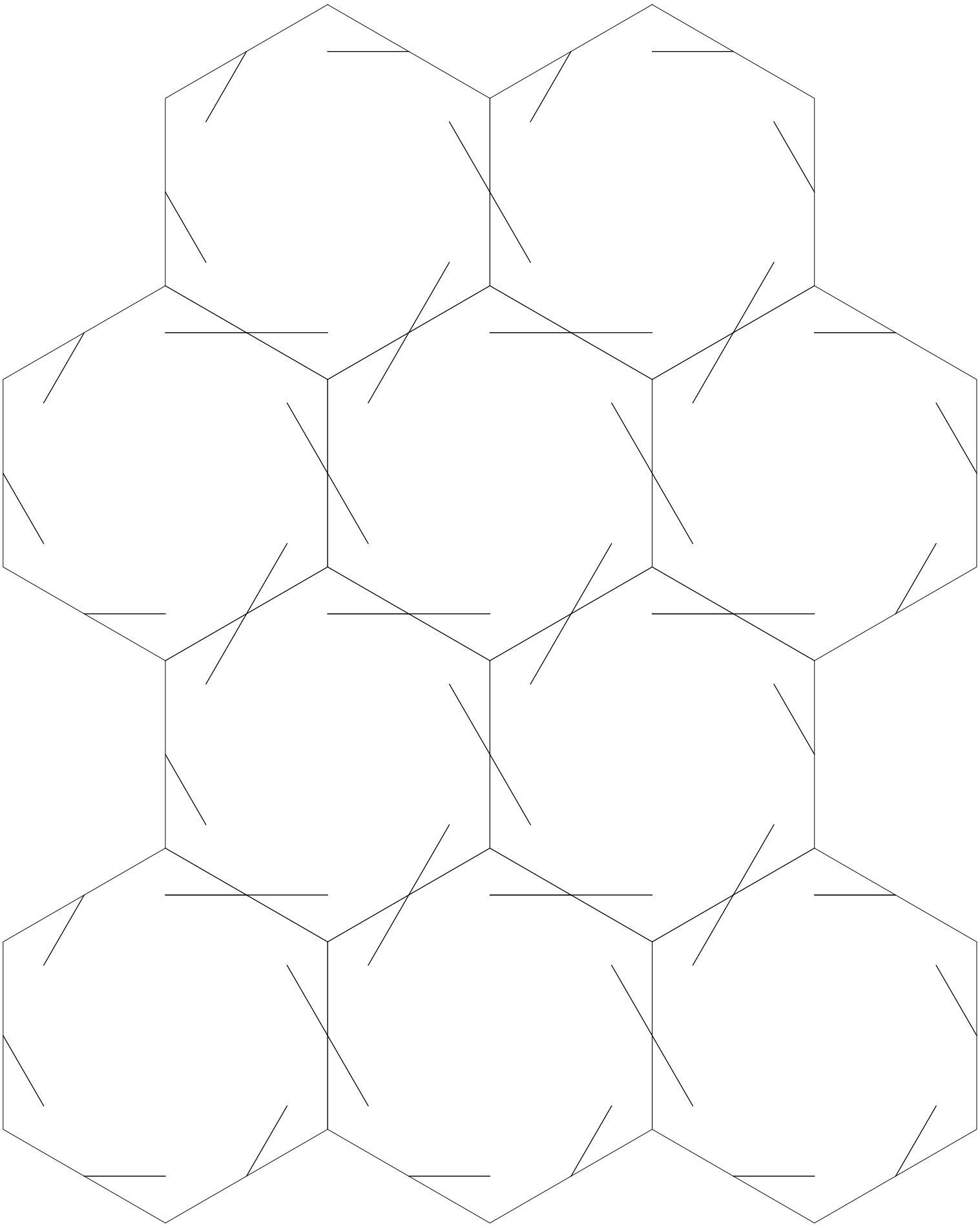
Rhombicuboctahedron 3/3



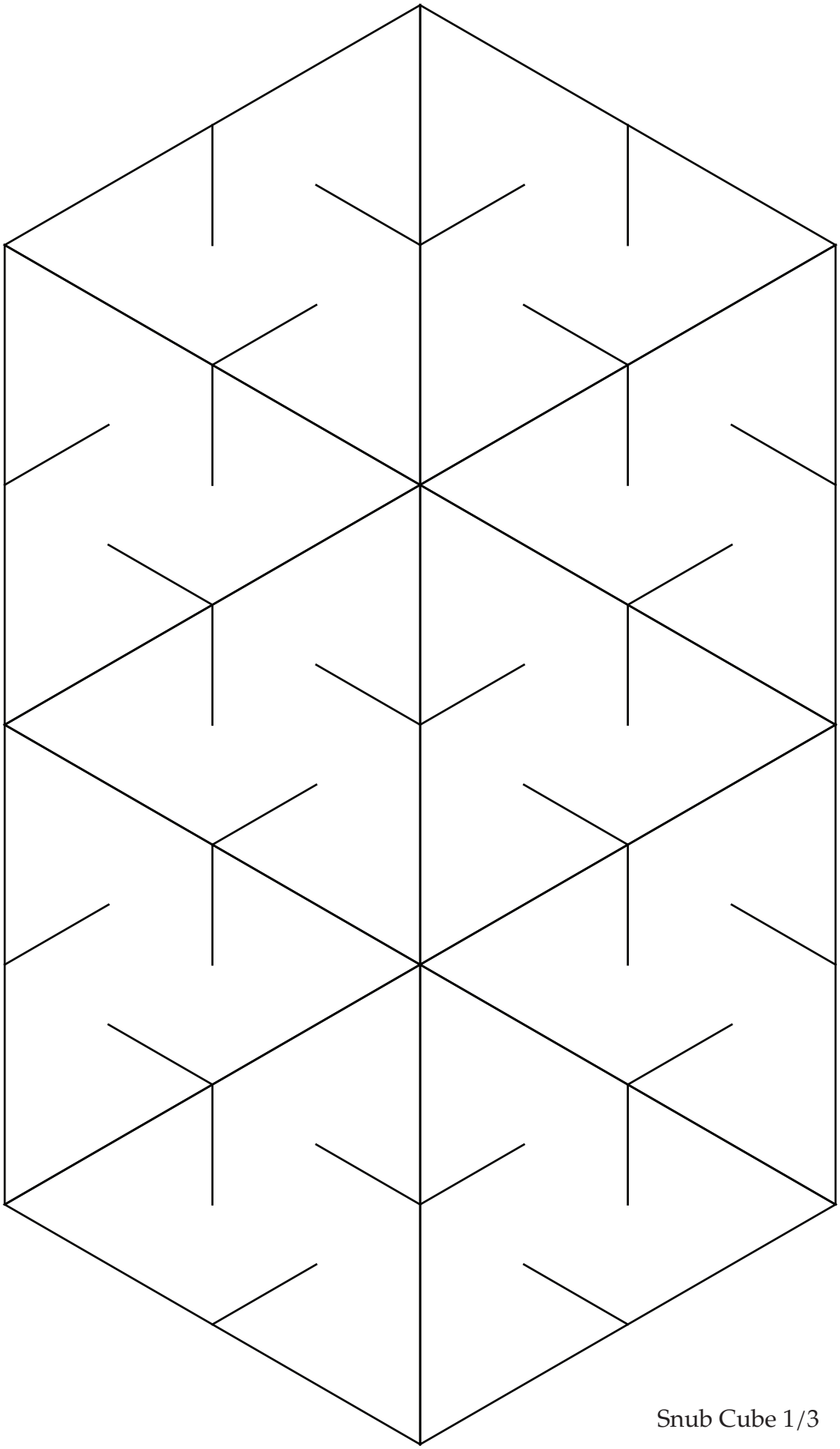
Truncated Icosahedron 1/3



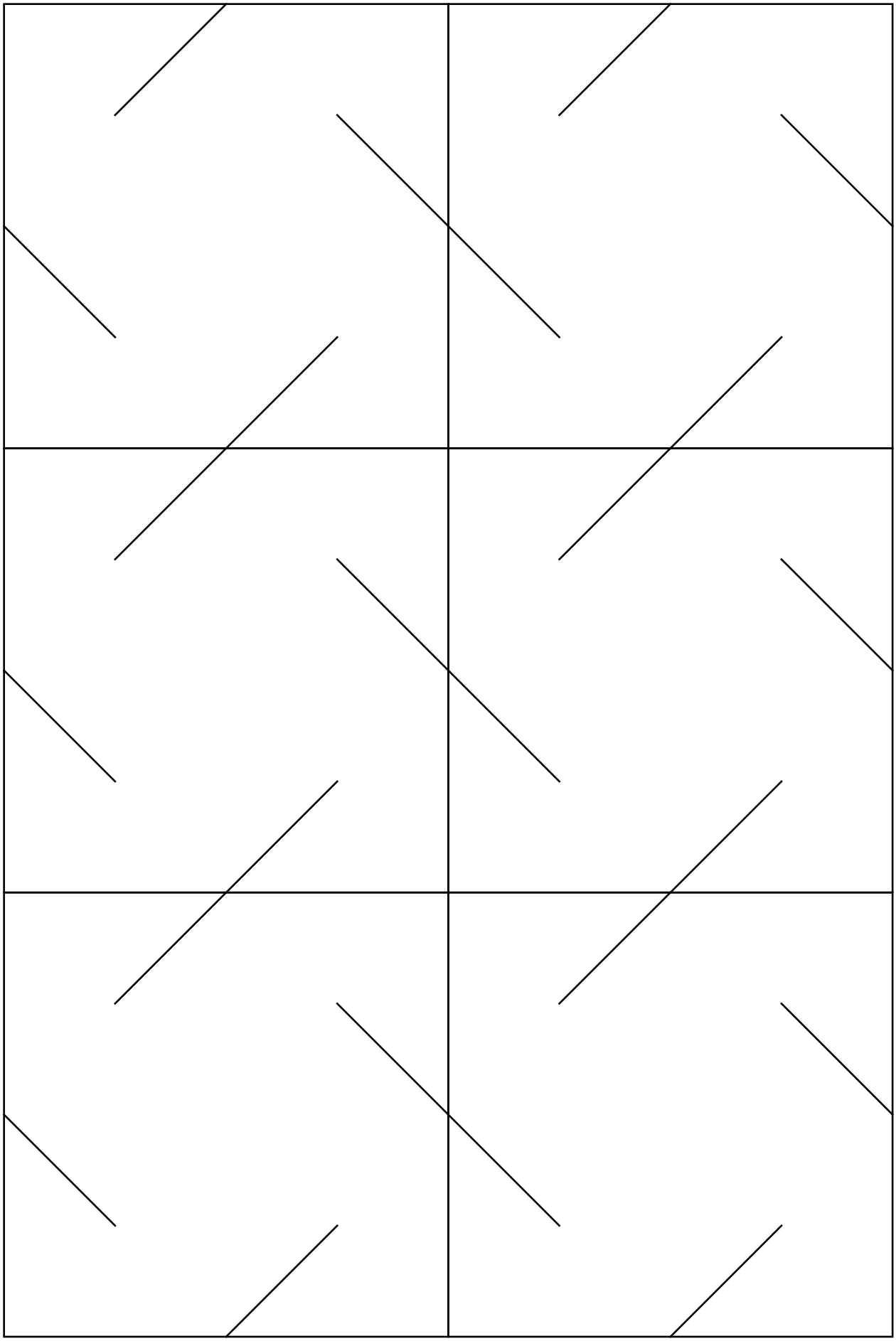
Truncated Icosahedron 2/3



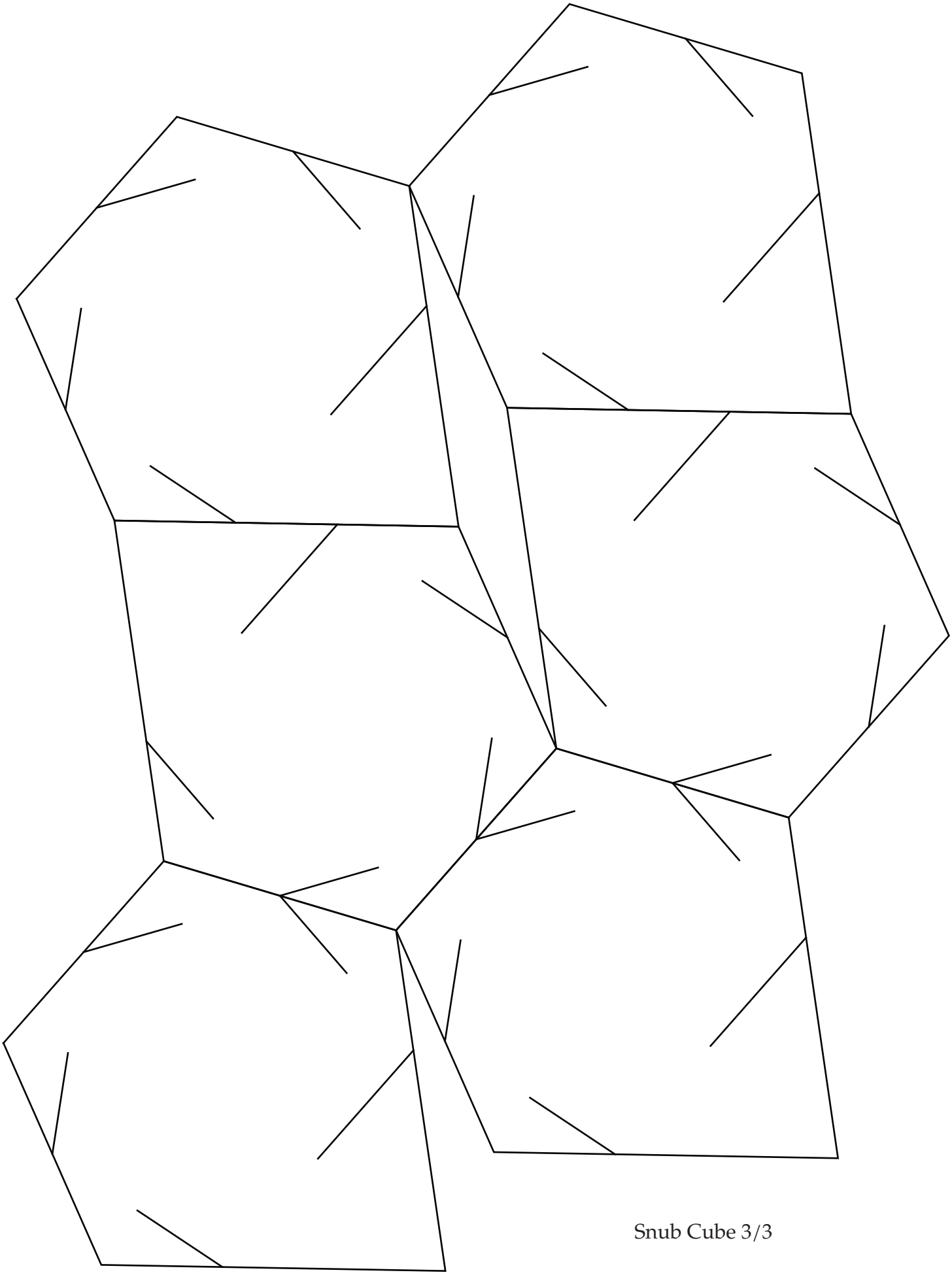
Truncated Icosahedron 3/3



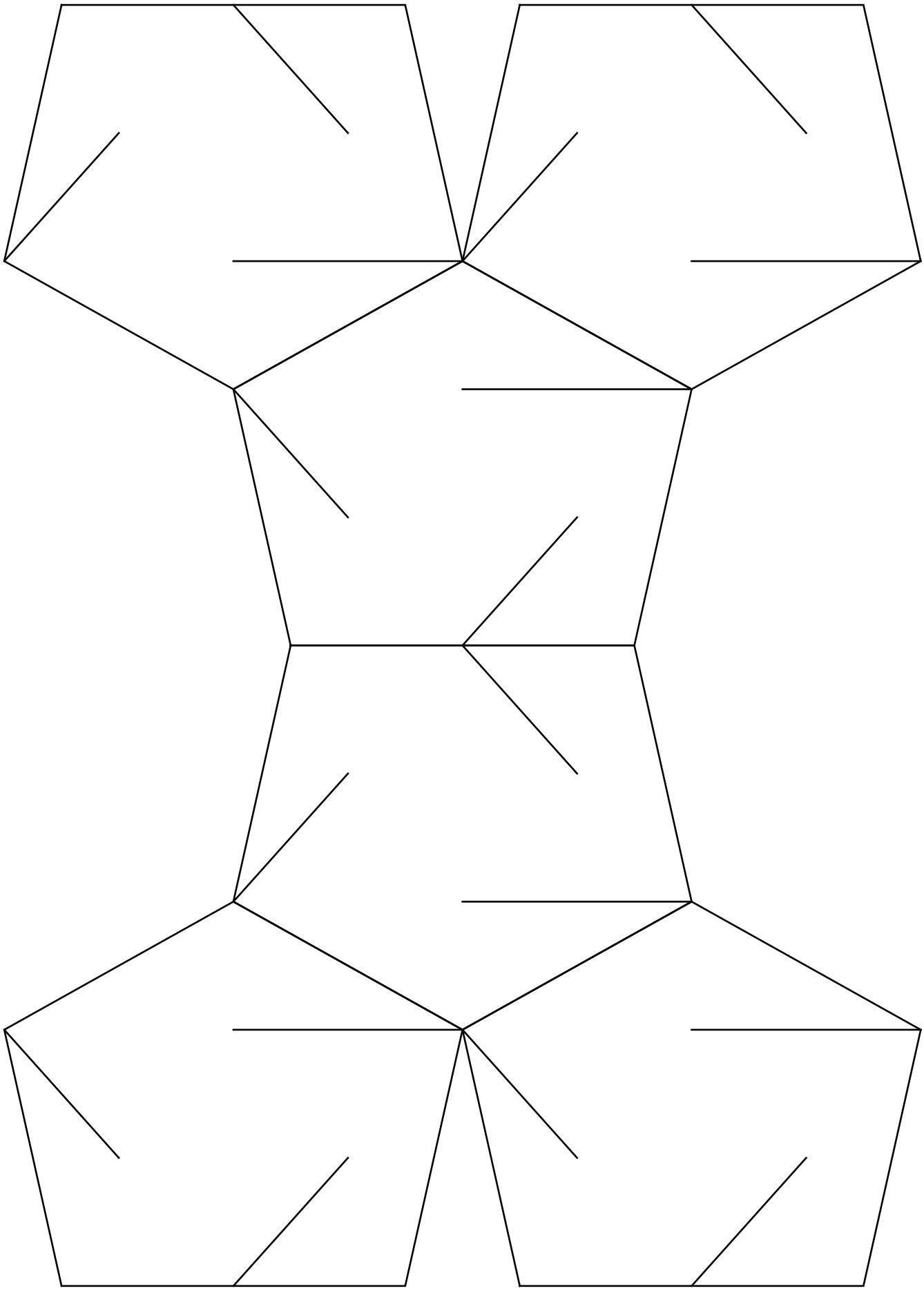
Snub Cube 1/3



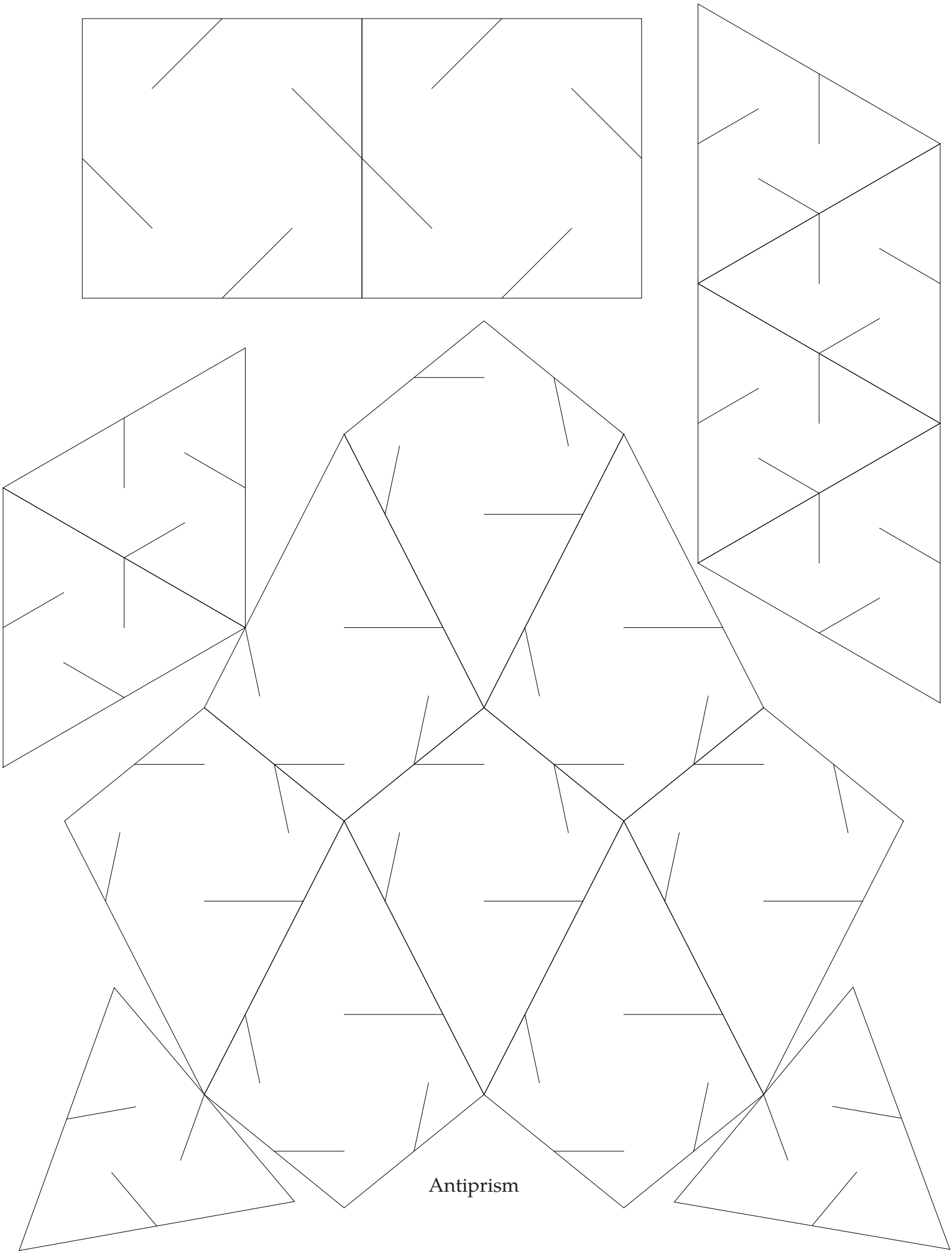
Snub Cube 2/3



Snub Cube 3/3



Pyritohedron



Antiprism